6.042/18.062J Mathematics for Computer Science — April 1, 2005

Srini Devadas and Eric Lehman

Notes for Recitation 14

Counting Rules

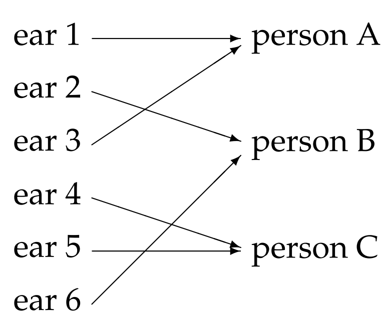
Rule 1 (Generalized Product Rule). *Let S be a set of length-k sequences. If there are:*

- *n_1 possible first entries,*
- *n_2 possible second entries for each first entry,*
- *n_3 possible third entries for each combination of first and second entries, etc.*

then:

$$|S| = n_1 \cdot n_2 \cdot n_3 \cdots n_k$$

A ***k-to-1 function*** maps exactly k elements of the domain to every element of the range. For example, the function mapping each ear to its owner is 2-to-1:



Rule 2 (Division Rule). *If $f : A \to B$ is k-to-1, then $|A| = k \cdot |B|$.*

The Generalized Product Rule

Problem 1. Solve the following counting problems using the generalized product rule.

(a) Next week, I'm going to get really fit! On day 1, I'll exercise for 5 minutes. On each subsequent day, I'll exercise 0, 1, 2, or 3 minutes more than the previous day. For example, the number of minutes that I exercise on the seven days of next week might be 5, 6, 9, 9, 9, 11, 12. How many such sequences are possible?

Solution. The number of minutes on the first day can be selected in 1 way. The number of minutes on each subsequent day can be selected in 4 ways. Therefore, the number of exercise sequences is $1 \cdot 4^6$ by the extended product rule.

(b) An r-***permutation*** of a set is a sequence of r distinct elements of that set. For example, here are all the 2-permutations of $\{a, b, c, d\}$:

$$\begin{array}{ccc} (a,b) & (a,c) & (a,d) \\ (b,a) & (b,c) & (b,d) \\ (c,a) & (c,b) & (c,d) \\ (d,a) & (d,b) & (d,c) \end{array}$$

How many r-permutations of an n-element set are there? Express your answer using factorial notation.

Solution. There are n ways to choose the first element, $n-1$ ways to choose the second, $n-2$ ways to choose the third, ..., and there are $n-r+1$ ways to choose the r-th element. Thus, there are:

$$n \cdot (n-1) \cdot (n-2) \cdots (n-r+1) \quad = \quad \frac{n!}{(n-r)!}$$

r-permutations of an n-element set.

(c) How many $n \times n$ matrices are there with *distinct* entries drawn from $\{1, \ldots, p\}$, where $p \geq n^2$?

Solution. There are p ways to choose the first entry, $p-1$ ways to choose the second for each way of choosing the first, $p-2$ ways of choosing the third, and so forth. In all there are

$$p(p-1)(p-2)\cdots(p-n^2+1) = \frac{p!}{(p-n^2)!}$$

such matrices. Alternatively, this is the number of n^2-permutations of a p element set, which is $p!/(p-n^2)!$.

The Tao of BOOKKEEPPER

Problem 2. In this problem, we seek enlightenment through contemplation of the word $BOOKKEEPER$.

(a) In how many ways can you arrange the letters in the word $POKE$?

Solution. There are $4!$ arrangments corresponding to the $4!$ permutations of the set $\{P, O, K, E\}$.

(b) In how many ways can you arrange the letters in the word BO_1O_2K? Observe that we have subscripted the O's to make them distinct symbols.

Solution. There are $4!$ arrangments corresponding to the $4!$ permutations of the set $\{B, O_1, O_2, K\}$.

(c) Suppose we map arrangements of the letters in BO_1O_2K to arrangements of the letters in $BOOK$ by erasing the subscripts. Indicate with arrows how the arrangements on the left are mapped to the arrangements on the right.

O_2BO_1K	
KO_2BO_1	$BOOK$
O_1BO_2K	$OBOK$
KO_1BO_2	$KOBO$
BO_1O_2K	$\ldots$
BO_2O_1K	
$\ldots$	

(d) What kind of mapping is this, young grasshopper?

Solution. 2-to-1

(e) In light of the Division Rule, how many arrangements are there of $BOOK$?

Solution. $4!/2$

(f) Very good, young master! How many arrangements are there of the letters in $KE_1E_2PE_3R$?

Solution. $6!$

(g) Suppose we map each arrangement of $KE_1E_2PE_3R$ to an arrangement of $KEEPER$ by erasing subscripts. List all the different arrangements of $KE_1E_2PE_3R$ that are mapped to $REPEEK$ in this way.

Solution. $RE_1PE_2E_3K$, $RE_1PE_3E_2K$, $RE_2PE_1E_3K$, $RE_2PE_3E_1K$, $RE_3PE_1E_2K$, $RE_3PE_2E_1K$

(h) What kind of mapping is this?

Solution. 3!-to-1

(i) So how many arrangements are there of the letters in $KEEPER$?

Solution. $6!/3!$

(j) *Now you are ready to face the BOOKKEEPER!*

How many arrangements of $BO_1O_2K_1K_2E_1E_2PE_3R$ are there?

Solution. $10!$

(k) How many arrangements of $BOOK_1K_2E_1E_2PE_3R$ are there?

Solution. $10!/2!$

(l) How many arrangements of $BOOKKE_1E_2PE_3R$ are there?

Solution. $10!/(2! \cdot 2!)$

(m) How many arrangements of $BOOKKEEPER$ are there?

Solution. $10!/(2! \cdot 2! \cdot 3!)$

(n) How many arrangements of $VOODOODOLL$ are there?

Solution. $10!/(2! \cdot 2! \cdot 5!)$

(o) **(IMPORTANT)** How many n-bit sequences contain k zeros and $(n-k)$ ones?

Solution. $n!/(k! \cdot (n-k)!)$

This quantity is denoted $\binom{n}{k}$ and read "n choose k". You will see it almost every day in 6.042 from now until the end of the term.

Remember well what you have learned: subscripts on, subscripts off.

This is the Tao of Bookkeeper.

Problem 3. Solve the following counting problems. Define an appropriate mapping (bijective or k-to-1) between a set whose size you know and the set in question.

(a) (IMPORTANT) In how many ways can k elements be chosen from an n-element set $\{x_1, x_2, \ldots, x_n\}$?

Solution. There is a bijection from n-bit sequences with k ones. The sequence $(b_1, \ldots, b_n)$ maps to the subset that contains x_i if and only if $b_i = 1$. Therefore, the number of such subsets is $\binom{n}{k}$.

(b) How many different ways are there to select a dozen donuts if four varieties are available?

Solution. There is a bijection from selections of a dozen donuts to 15-bit sequences with exactly 3 ones. In particular, suppose that the varieties are glazed, chocolate, lemon, and Boston creme. Then a selection of g glazed, c chocolate, l lemon, and b Boston creme maps to the sequence:

$$(g\ 0's)\ 1\ (c\ 0's)\ 1\ (l\ 0's)\ 1\ (b\ 0's)$$

Therefore, the number of selections is equal to the number of 15-bit sequences with exactly 3 ones, which is:

$$\frac{15!}{3!\ 12!} = \binom{15}{3}$$

(c) How many paths are there from $(0, 0)$ to $(10, 20)$ consisting of right-steps (which increment the first coordinate) and up-steps (which increment the second coordinate)?

Solution. There is a bijection from 30-bit sequences with 10 zeros and 20 ones. The sequence $(b_1, \ldots, b_{30})$ maps to a path where the i-th step is right if $b_i = 0$ and up if $b_i = 1$. Therefore, the number of paths is equal to $\binom{30}{10}$.

(d) An independent living group is hosting eight pre-frosh, affectionately known at $P_1, \ldots, P_8$ by the permanent residents. Each pre-frosh must must be assigned a task: 2 must wash pots, 2 must clean the kitchen, 1 must clean the bathrooms, 1 must clean the common area, and 2 must serve dinner. In how many ways can $P_1, \ldots, P_8$ be put to productive use?

Solution. There is a bijection from sequences containing two P's, two K's, a B, a C, and two D's. In particular, the sequence $(t_1, \ldots, t_8)$ corresponds to assigning P_i to washing pots if $t_i = P$, to cleaning the kitchen if $t_i = K$, to cleaning the bathrooms if $t_i = B$, etc. Therefore, the number of possible assignments is:

$$\frac{8!}{2!\ 2!\ 1!\ 1!\ 2!}$$

(e) In how many ways can Mr. and Mrs. Grumperson distribute 13 indistinguishable pieces of coal to their two— no, three!— children for Christmas?

Solution. There is a bijection from 15-bit strings with two ones. In particular, the bit string $0^a 1 0^b 1 0^c$ maps to the assignment of a coals to the first child, b coals to the second, and c coals to the third. Therefore, there are $\binom{15}{2}$ assignments.

(f) How many solutions over the natural numbers are there to the equation:

$$x_1 + x_2 + \ldots + x_{10} \quad \leq \quad 100$$

Solution. There is a bijection from 110-bit sequences with 10 ones to solutions to this equation. In particular, x_i is the number of zeros before the i-th one but after the $(i-1)$-st one (or the beginning of the sequence). Therefore, there are $\binom{110}{10}$ solutions.

(g) (Quiz 2, Fall '03) Suppose that two identical 52-card decks of are mixed together. In how many ways can the cards in this double-size deck be arranged?

Solution. The number of sequences of the 104 cards containing 2 of each card is $104!/(2!)^{52}$.