Notes for Recitation 12

1 Solving linear recurrences

Guessing a *particular solution*. Recall that a general linear recurrence has the form:

$$f(n) = a_1 f(n-1) + a_2 f(n-2) + \cdots + a_d f(n-d) + g(n)$$

As explained in lecture, one step in solving this recurrence is finding a *particular solution*; i.e., a function $f(n)$ that satisfies the recurrence, but may not be consistent with the boundary conditions. Here's a recipe to help you guess a particular solution:

- If $g(n)$ is a constant, guess that $f(n)$ is some constant c. Plug this into the recurrence and see if any constant actually works. If not, try $f(n) = bn + c$, then $f(n) = an^2 + bn + c$, etc.
- More generally, if $g(n)$ is a polynomial, try a polynomial of the same degree. If that fails, try a polynomial of degree one higher, then two higher, etc. For example, if $g(n) = n$, then try $f(n) = bn + c$ and then $f(n) = an^2 + bn + c$.
- If $g(n)$ is an exponential, such as 3^n, then first guess that $f(n) = c3^n$. Failing that, try $f(n) = bn3^n + c3^n$ and then $an^2 3^n + bn3^n + c3^n$, etc.

In practice, your first or second guess will almost always work.

Dealing with *repeated roots*. In lecture we saw that the solutions to a linear recurrence are determined by the roots of the characteristic equation: For each root r of the equation,

the function r^n is a solution to the recurrence.

Taking a linear combination of these solutions, we can move on to find the coefficients.

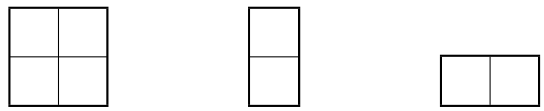
The situation is a little more complicated when r is a *repeated root* of the characteristic equation: if its multiplicity is k, then (not only r^n, but)

each of the functions $r^n, \; nr^n, \; n^2r^n, \; \ldots, \; n^{k-1}r^n$ is a solution to the recurrence,

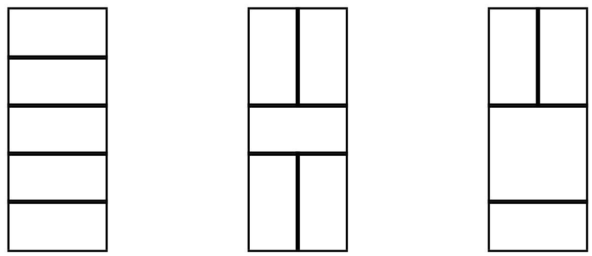
so that our linear combination must use all of them.

2 Mini-Tetris (again)

Remember Mini-Tetris from Recitation 4? Here is an overview: A *winning configuration* in the game is a complete tiling of a $2 \times n$ board using only the three shapes shown below:



For example, the several possible winning configurations on a 2×5 board include:



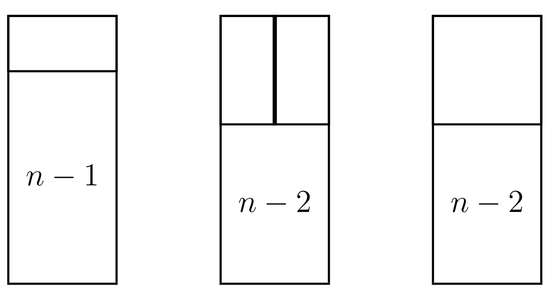
In that past recitation, we had defined T_n to be the number of different winning configurations on a $2 \times n$ board. Then we had to inductively prove T_n equals some particular closed form expression. Remember that expression? Probably not. But no damage, now you can find it on your own.

(a) Determine the values of T_1, T_2, and T_3.

Solution. $T_1 = 1$, $T_2 = 3$, and $T_3 = 5$.

(b) Find a recurrence equation that expresses T_n in terms of T_{n-1} and T_{n-2}.

Solution. Every winning configuration on a $2 \times n$ board is of one three types, distinguished by the arrangment of pieces at the top of the board.



There are T_{n-1} winning configurations of the first type, and there are T_{n-2} winning configurations of each of the second and third types. Overall, the number of winning configurations on a $2 \times n$ board is:

$$T_n = T_{n-1} + 2T_{n-2}$$

(c) Find a closed-form expression for T_n.

Solution. The characteristic polynomial is $r^2 - r - 2 = (r-2)(r+1)$, so the solution is of the form $A2^n + B(-1)^n$. Setting $n = 1$, we have $1 = T_1 = 2A - B$. Setting $n = 2$, we have $3 = T_2 = A2^2 + B(-1)^2 = 4A + B$. Solving these two equations, we conclude $A = 2/3$ and $B = 1/3$. That is, the closed form expression for T_n is

$$T_n = \frac{2}{3}2^n + \frac{1}{3}(-1)^n = \frac{2^{n+1} + (-1)^n}{3}.$$

Remember it now?

3 Inhomogeneous linear recurrences

Find a closed-form solution to the following linear recurrence.

$$\begin{aligned} T_0 &= 0 \\ T_1 &= 1 \\ T_n &= T_{n-1} + T_{n-2} + 1 \qquad (*) \end{aligned}$$

(a) First find the general solution to the corresponding homogenous recurrence.

Solution. The characteristic equation is $r^2 - r - 1 = 0$. The roots of this equation are:

$$\begin{aligned} r_1 &= \frac{1+\sqrt{5}}{2} \\ r_2 &= \frac{1-\sqrt{5}}{2} \end{aligned}$$

Therefore, the solution to the homogenous recurrence is of the form

$$T_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

(b) Now find a particular solution to the inhomogenous recurrence.

Solution. Since the inhomogenous term is constant, we guess a constant solution, c. So replacing the T terms in (*) by c, we require

$$c = c + c + 1,$$

namely, $c = -1$. That is, $T_n \equiv -1$ is a particular solution to (*).

(c) The complete solution to the recurrence is the homogenous solution plus the particular solution. Use the initial conditions to find the coefficients.

Solution.

$$T_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n - 1$$

All that remains is to find the constants A and B. Substituting the initial conditions gives a system of linear equations.

$$\begin{aligned} 0 &= A + B - 1 \\ 1 &= A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right) - 1 \end{aligned}$$

The solution to this linear system is:

$$\begin{aligned} A &= \frac{5+3\sqrt{5}}{10} \\ B &= \frac{5-3\sqrt{5}}{10} \end{aligned}$$

(d) Therefore, the complete solution to the recurrence is:

Solution.

$$T_n = \left(\frac{5+3\sqrt{5}}{10}\right)\cdot\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{5-3\sqrt{5}}{10}\right)\cdot\left(\frac{1-\sqrt{5}}{2}\right)^n - 1.$$

4 Back to homogeneous ones

Let's get back to homogeneous linear recurrences. Find a closed-form solution to this one.

$$\begin{aligned} S_0 &= 0 \\ S_1 &= 1 \\ S_n &= 6S_{n-1} - 9S_{n-2} \end{aligned}$$

Anything strange?

Solution. The characteristic polynomial is $r^2 - 6r + 9 = (r-3)^2$, so we have a *repeated root*: $r = 3$, with multiplicity 2. The solution is of the form $A3^n + Bn3^n$ for some constants A and B. Setting $n = 0$, we have $0 = S_0 = A3^0 + B \cdot 0 \cdot 3^0 = A$. Setting $n = 1$, we have $1 = S_1 = A3^1 + B \cdot 1 \cdot 3^1 = 3B$, so $B = 1/3$. That is,

$$S_n = 0 \cdot 3^n + \frac{1}{3} \cdot n3^n = n3^{n-1}.$$

Short Guide to Solving Linear Recurrences

A *linear recurrence* is an equation

$$\underbrace{f(n) = a_1 f(n-1) + a_2 f(n-2) + \ldots + a_d f(n-d)}_{\text{homogeneous part}} \qquad \underbrace{+g(n)}_{\text{inhomogeneous part}}$$

together with boundary conditions such as $f(0) = b_0$, $f(1) = b_1$, etc.

1. Find the roots of the *characteristic equation*:

$$x^n = a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_k$$

2. Write down the *homogeneous solution*. Each root generates one term and the homogeneous solution is the sum of these terms. A nonrepeated root r generates the term $c_r r^n$, where c_r is a constant to be determined later. A root r with multiplicity k generates the terms:

$$c_{r_1} r^n, \quad c_{r_2} n r^n, \quad c_{r_3} n^2 r^n, \quad \ldots, \quad c_{r_k} n^{k-1} r^n$$

   where $c_{r_1}, \ldots, c_{r_k}$ are constants to be determined later.

3. Find a *particular solution*. This is a solution to the full recurrence that need not be consistent with the boundary conditions. Use guess and verify. If $g(n)$ is a polynomial, try a polynomial of the same degree, then a polynomial of degree one higher, then two higher, etc. For example, if $g(n) = n$, then try $f(n) = bn + c$ and then $f(n) = an^2 + bn + c$. If $g(n)$ is an exponential, such as 3^n, then first guess that $f(n) = c3^n$. Failing that, try $f(n) = bn3^n + c3^n$ and then $an^2 3^n + bn3^n + c3^n$, etc.

4. Form the *general solution*, which is the sum of the homogeneous solution and the particular solution. Here is a typical general solution:

$$f(n) = \underbrace{c2^n + d(-1)^n}_{\text{homogeneous solution}} + \underbrace{3n + 1}_{\text{particular solution}}$$

5. Substitute the boundary conditions into the general solution. Each boundary condition gives a linear equation in the unknown constants. For example, substituting $f(1) = 2$ into the general solution above gives:

$$\begin{aligned} 2 &= c \cdot 2^1 + d \cdot (-1)^1 + 3 \cdot 1 + 1 \\ \Rightarrow \quad -2 &= 2c - d \end{aligned}$$

   Determine the values of these constants by solving the resulting system of linear equations.