Quiz 2

YOUR NAME: ____________________________

- **Calculators are not allowed on this exam.**
- You may use one $8.5 \times 11''$ sheet with notes in your own handwriting on both sides, but no other sources of information.
- You may assume all results from lecture, the notes, problem sets, and recitation.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem.
- Be neat and write legibly. You will be graded not only on the correctness of your answers, but also on the clarity with which you express them.
- The exam ends at 9:30 PM.
- GOOD LUCK!

Problem	Points	Grade	Grader
1	10		
2	10		
3	15		
4	15		
5	20		
6	15		
7	15		
Total	100		

NOTE: For this exam, a "closed form" is a mathematical expression *without* summation notation, product notation, or the ... symbol. Factorials and binomial coefficients *may* appear in a closed form. Some examples are shown below.

Closed Forms	NOT Closed Forms
42	$\sum_{k=0}^{n} k^2$
$(\sqrt{x}+1)^n$	$\prod_{i=1}^{n} \left(1+\frac{1}{i}\right)$
$n! + \binom{n}{3}$	$\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}$

Problem 1. [10 points] Let S consist of all positive integers with no prime factor larger than 3, and define:

$$X = \sum_{k \in S} \frac{1}{k}$$

Thus, the first few terms of the sum are:

$$X = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} + \frac{1}{12} + \ldots$$

(a) Write a closed-form expression in the box that makes the equation below true.

$$X = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \boxed{}$$

Solution. Every positive integer with no prime factor larger than 3 has the form $2^j 3^k$ for some nonnegative integers j and k. Thus, the expression

$$\frac{1}{2^j 3^k}$$

makes the equation true.

(b) Write a closed-form expression in the box that makes this equation true:

$$X = \boxed{}$$

Solution. We'll apply the formula for an infinite geometric sum twice.

$$\begin{aligned}
\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{2^j 3^k} &= \sum_{j=0}^{\infty} \frac{1}{2^j} \left(\sum_{k=0}^{\infty} \frac{1}{3^k} \right) \\
&= \sum_{j=0}^{\infty} \frac{1}{2^j} \left(\frac{1}{1 - 1/3} \right) \\
&= \left(\frac{1}{1 - 1/3} \right) \sum_{j=0}^{\infty} \frac{1}{2^j} \\
&= \left(\frac{1}{1 - 1/3} \right) \left(\frac{1}{1 - 1/2} \right) \\
&= 3
\end{aligned}$$

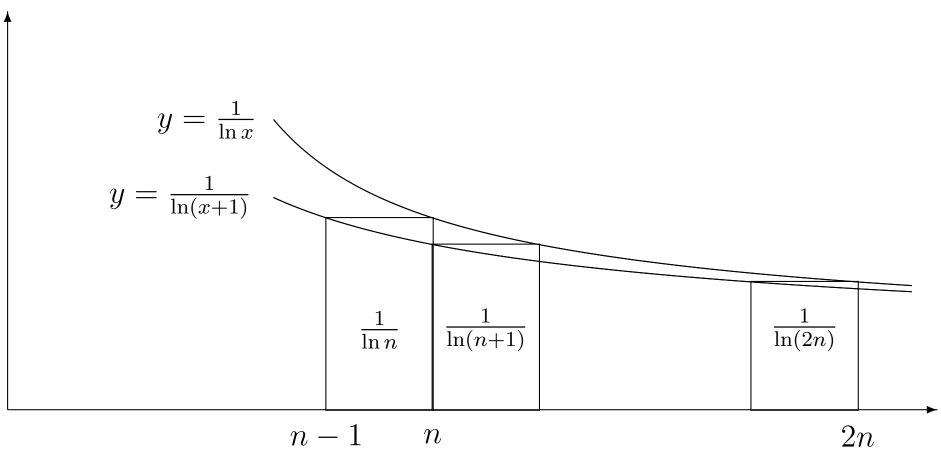
Problem 2. [10 points] Derive integrals that are closely-matching lower and upper bounds on the sum

$$\sum_{k=n}^{2n} \frac{1}{\ln k}$$

where $n \geq 3$. Justify your answers with a diagram. **Do not integrate.** Your answers should be unevaluated integrals.

(a) Draw your diagram in the space below. (To receive full credit, the diagram must clearly communicate why your integral bounds are correct.)

Solution.



(b) Write your lower bound integral here $\longrightarrow$ $\int_{n-1}^{2n} \frac{1}{\ln(x+1)}\, dx$

(c) Write your upper-bound integral here $\longrightarrow$ $\int_{n-1}^{2n} \frac{1}{\ln x}\, dx$

Problem 3. [15 points] Solve the following problems involving asymptotic notation. Here H_n is the n-th harmonic number; thus, $H_n = 1/1 + 1/2 + \ldots + 1/n$.

(a) Circle **all symbols** on the right that could properly appear in the box on the same line. (There may be more than one!)

$$n^2 = \square \left(\frac{n^2 \log n}{\sqrt{n+1}}\right) \qquad \Theta \quad O \quad \Omega \quad o$$

$$2^n = \square \left(3^n - n^3\right) \qquad \Theta \quad O \quad \Omega \quad o$$

$$2H_n = \square (\ln n) \qquad \Theta \quad O \quad \Omega \quad o$$

$$n^{0.01} = \square \left((\ln n)^{100}\right) \qquad \Theta \quad O \quad \Omega \quad o$$

Solution. (1) Ω (2) O, o (3) Θ, O, Ω (4) Ω (5) Ω.

(b) Suppose that $f(n) \sim g(n)$. Beside each statement below that *must* be true, circle **true**. Beside the remaining statements, circle **false**.

$f(n)^2 \sim g(n)^2$	**true**	**false**
$f(n) = O(g(n))$	**true**	**false**
$f(n) = o(g(n))$	**true**	**false**
$2^{f(n)} = \Theta(2^{g(n)})$	**true**	**false**

Solution. (a) True. (b) True. (c) False for all f, g. (d) False. Let $f(n) = n$, $g(n) = n + \log n$.

Problem 4. [15 points] A misguided MIT student designs a self-replicating 6.270 robot. The student builds one such robot every day, starting on day 0. The day after a robot is built, it constructs two copies of itself. (On all subsequent days, the robot busily searches for ping-pong balls— these are 6.270 robots, after all.) Here is what happens over the first few days:

Day 0. The student builds robot R_1.

Day 1. The student builds robot R_2. Robot R_1 builds robots R_3 and R_4.

Day 2. The student builds R_5. Robot R_2 builds R_6 and R_7, robot R_3 builds R_8 and R_9, and robot R_4 builds R_{10} and R_{11}. Robot R_1 searches for ping-pong balls.

Day 3. The student builds R_{12}. Robots $R_5, \ldots, R_{11}$ build robots $R_{13}, \ldots, R_{26}$. Robots R_1, R_2, R_3, and R_4 search for ping-pong balls.

Let T_n be the number of robots in existence at the end of day n. Thus, $T_0 = 1$, $T_1 = 4$, $T_2 = 11$, and $T_3 = 26$.

(a) How many new robots are built on day $n-1$? Express your answer in terms of the variables $T_{n-1}, T_{n-2}, \ldots$ and assume $n \geq 2$.

Solution. This is the difference between the number that existed on day $n-1$ and the number that existed on day $n-2$, which is $T_{n-1} - T_{n-2}$.

(b) Express T_n with a recurrence equation and sufficient base cases. **Do not solve the recurrence.**

Solution. The number of robots on day n is equal to the number of robots on day $n-1$, plus twice the number of robots built yesterday ($T_{n-1} - T_{n-2}$), plus the 1 robot built by the student. Therefore, we have:

$$
\begin{aligned}
T_0 &= 1 \\
T_1 &= 4 \\
T_n &= 3T_{n-1} - 2T_{n-2} + 1 \quad (\text{for } n \geq 2)
\end{aligned}
$$

(c) An even more misguided 6.270 student designs another self-replicating robot to hunt down and destroy robots of the first kind. The number of these robots at the end of day n is P_n, where:

$$\begin{aligned} P_0 &= 0 \\ P_1 &= 1 \\ P_n &= 5P_{n-1} - 6P_{n-2} + 1 \quad (\text{for } n \geq 2) \end{aligned}$$

Find a closed-form expression for P_n. Show your work clearly to be eligible for partial credit.

Solution. The characteristic equation is $x^2 - 5x + 6 = 0$. The right side factors into $(x - 2)(x - 3)$, so the roots are 2 and 3. For a particular solution, let's first guess $P_n = c$. Substituting this into the recurrence equation gives $c = 5c - 6c + 1$, which implies that $c = 1/2$. Therefore, the general form of the solution is:

$$P_n = A \cdot 2^n + B \cdot 3^n + 1/2$$

Substituting $P_0 = 0$ and $P_1 = 1$ gives the equations:

$$\begin{aligned} 0 &= A + B + 1/2 \\ 1 &= 2A + 3B + 1/2 \end{aligned}$$

Solving this system gives $A = -2$ and $B = 3/2$. Therefore, the solution is:

$$P_n = -2 \cdot 2^n + \frac{3}{2} \cdot 3^n + 1/2$$

Problem 5. [20 points] Solve the following counting problems. Your answers must be closed forms, but need not be simplified. In particular, you may leave factorials and binomial coefficients in your answers. To be eligible for partial credit, you must explain how you arrived at your answer.

(a) Four card players (Alice, Bob, Carol, and Dave) are each dealt a 7-card hand from a 52-card deck. In how many different ways can this be done?

Solution. There is a bijection with 52-symbol sequences containing 7 A's, 7 B's, 7 C's, 7 D's and 24 X's (indicating cards that remain in the deck). Thus, the number of ways to deal out the cards is

$$\frac{51!}{7!^4\, 24!}$$

by the Bookkeeper Rule.

(b) Stinky Peterson has decided to start a Bug Farm under his bed. He plans to raise 100 bugs selected from four basic varieties: *creepy*, *crawly*, *fuzzy*, and *slimey*. Assuming he wants at least 10 speciments of each, how many different distributions are possible? (For example, one possible distribution is 20 creepy, 20 crawly, 10 fuzzy, and 50 slimey.)

Solution. First, he places 10 specimens of each under his bed. Then he must select the remaining 60 additional speciments from the four kinds of bug. There is a bijection between such selections and 63-bit sequences with exactly 3 ones, so the number of distributions is

$$\frac{63!}{60!\, 3!} = \binom{63}{60}$$

by the Bookkeeper Rule.

(c) There are n runners in a race. Before the race, each runner is assigned a number between 1 and n. The runners can finish the race in any one of $n!$ different orders. In how many of these orders is the first finisher not #1, the second finisher not #2, and the third finisher not #3?

Solution. Let P_k be the set of finishing orders in which runner $\#k$ *is* is the k-th finisher. In these terms, the solution is:

$$\begin{aligned} n! - |P_1 \cup P_2 \cup P_3| &= n! - (|P_1| + |P_2| + |P_3| \\ &\qquad - |P_1 \cap P_2| - |P_1 \cap P_3| - |P_2 \cap P_3| \\ &\qquad + |P_1 \cap P_2 \cap P_3|) \\ &= n! - 3(n-1)! + 3(n-2)! - (n-3)! \end{aligned}$$

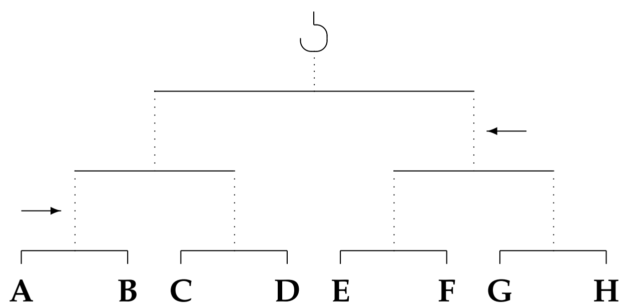
(d) How many ways are there to park 4 identical SUVs and 10 identical cars in a row of 20 parking spaces if SUVs are too wide to park next to each other? For example, here is one parking possibility:

SUV		car	car			car	SUV	car	SUV		car	car		car		car	SUV	car	car

Solution. First, let's park the SUVs. The number of ways to do this is equal to the number of ways to select 20 books off of a shelf such that adjacent books are not selected— a problem of a type you've seen before. The answer is $\binom{17}{4}$. Now the 10 cars can be parked in the 16 remaining spaces in $\binom{16}{10}$ ways. So the total number of parking possibilities is:

$$\binom{17}{4} \cdot \binom{16}{10}$$

(e) A *mobile* is a hanging structure built from seven horizontal rods (indicated with solid lines), seven vertical strings (indicated with dotted lines), and eight toys (indicated with the letters A-H).



Many different toy arrangements can be obtained by twisting the strings. For example, twisting the string marked with the → arrow would swap toys **A** and **B**. Twisting the string marked with the ← arrow would *reverse* the order of toys **E**, **F**, **G**, and **H**. On the other hand, no combination of twists swaps only toys **B** and **C**. Two mobiles are different if one can not be obtained from the other by twisting strings. How many different mobiles are possible?

Solution. There are $8!$ different sequences of toys. Each mobile can be configured in 2^7 different ways, by twisting or not twisting the 7 upper strings. Thus, there is a 2^7-to-1 mapping from sequences to mobiles. By the Division Rule, the number of different mobiles is $8!/2^7 = 315$.

Problem 6. [15 points] A *subsequence* is obtained from a sequence by deleting one or more terms. For example, the sequence $(1, 2, 3, 4, 5)$ contains $(2, 4, 5)$ as a subsequence.

Theorem. *Every sequence of* $n^2 + 1$ *distinct integers contains an increasing or decreasing subsequence of length* $n + 1$.

For example, in the $3^2+1 = 10$ term sequence $(5, 6, \underline{1}, \underline{4}, 9, 0, 2, \underline{7}, \underline{8}, 3)$, the underlined terms form an increasing sequence of length $3 + 1 = 4$. Fill in the outline of a proof provided below.

Proof.

(a) Label each term in the sequence with the length of the longest increasing subsequence that ends with that term. For example, here is a sequence with the corresponding labels listed below.

(2,	6,	1,	5,	4,	9,	0,	8,	3,	7)
1	2	1							

Solution.

$$1, 2, 1, 2, 2, 3, 1, 3, 2, 3$$

(b) Now there are two cases. If some term is labeled $n+1$ or higher, then the theorem is true because

Solution. this means there is an increasing sequence of length at least $n + 1$ that ends with that term.

(c) Otherwise, at least $n + 1$ terms must have the same label $b \in \{1, 2, \ldots, n\}$ because

Solution. of the Pigeonhole Principle. Regard the labels a pigeons and the numbers $1, 2, \ldots, n$ as pigeonholes. Assign each label to its value. Since there are $n^2 + 1$ pigeons and only n pigeonholes, some $n + 1$ pigeons must be assigned to some hole b.

(d) The theorem is also true in this case because

Solution. Each of the $n + 1$ terms labeled b must be smaller than the one before. (Otherwise, a term labeled b could be appended to the length-b increasing sequence that ends at its predecessor to obtain a length-$(b + 1)$ increasing sequence. But then the term should actually be labeled $b + 1$.) Therefore, the $n + 1$ terms labeled b form a decreasing subsequence.

$\square$

Problem 7. [15 points] Every day in the life of Dangerous Dan is a potential disaster:

- Dan may or may not spill his breakfast cereal on his computer keyboard.
- Dan may or may not fall down the front steps on his way out the door.
- Dan stubs his toe zero or more times.
- Dan blurts something foolish an even number of times.

Let T_n be the number of different combinations of n mishaps Dan can suffer in one day. For example, $T_3 = 7$, because there are seven possible combinations of three mishaps:

spills	0	1	0	1	1	0	0
falls	0	0	1	1	0	1	0
stubs	3	2	2	1	0	0	1
blurts	0	0	0	0	2	2	2

(a) Give a generating function $g(x)$ for the sequence $\{T_0, T_1, T_2, \ldots\}$.

Solution. We multiply the generating functions for spills $(1+x)$, falls $(1+x)$, stubs $(1 + x + x^2 + \ldots = 1/(1-x))$, and blurts $(1 + x^2 + x^4 + \ldots = 1/(1-x^2))$:

$$\frac{(1+x)^2}{(1-x)(1-x^2)} = \frac{1+x}{(1-x)^2}$$

(b) Put integers in the boxes that make this equation true:

$$g(x) = \frac{\boxed{}}{1-x} + \frac{\boxed{}}{(1-x)^2}$$

Solution. $-1, 2$

(c) Put a closed-form expression in the box that makes this equation true:

$$T_n = \boxed{}$$

Remember that $1/(1-x)^2$ generates the sequence $\langle 1, 2, 3, \ldots \rangle$.

Solution. $2(n+1) - 1 = 2n + 1$