In-Class Problems Week 2, Fri.

Problem 1. Subset take-away[1] is a two player game involving a fixed finite set, A. Players alternately choose proper, nonempty subsets of A with the condition that one may not name a set containing a set that was named earlier. A player who is unable to move loses.

For example, if A is $\{1\}$, then there are no legal moves and the second player wins. If A is $\{1, 2\}$, then the only legal moves are $\{1\}$ and $\{2\}$. Each is a good reply to the other, and so once again the second player wins.

The first interesting case is when A has three elements. This time, if the first player picks a subset with one element, the second player picks the subset with the other two elements. If the first player picks a subset with two elements, the second player picks the subset whose sole member is the third element. Both cases produce positions equivalent to the starting position when A has two elements, and thus leads to a win for the second player.

Verify that when A has four elements, the second player still has a winning strategy.[2]

Problem 2. **(a)** Define a bijection between $\mathbb{N}$ and $\mathbb{Z}$.

(b) Define a bijection between $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$ (the ordered pairs (0,0),(0,1),(1,2),... of natural numbers).

[1] From Christenson & Tilford, *David Gale's Subset Takeaway Game, American Mathematical Monthly, Oct. 1997*

[2] David Gale worked out some of the propeeties of this game and made conjectured that the second player wins the game for any set A. This remains an open problem.