Solutions to Problem Set 9

Problem 1. Professor Plum, Mr. Green, and Miss Scarlet are all plotting to shoot Colonel Mustard. If one of these three has both an *opportunity* and the *revolver*, then that person shoots Colonel Mustard. Otherwise, Colonel Mustard escapes. Exactly one of the three has an *opportunity* with the following probabilities:

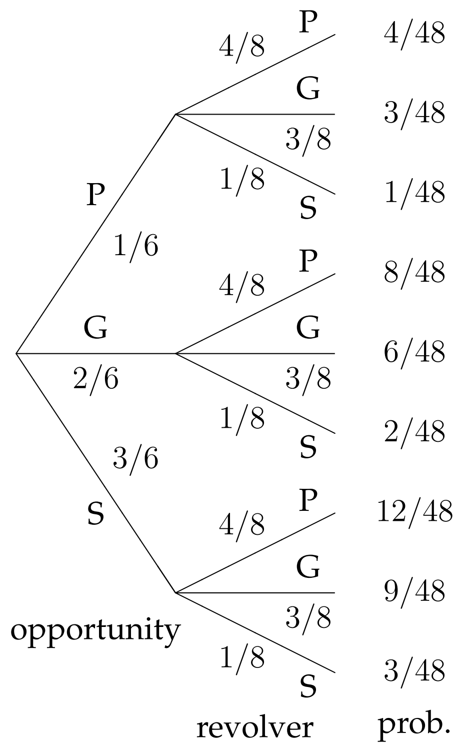
$$\begin{aligned}
\Pr\{\text{Plum has opportunity}\} &= 1/6\\
\Pr\{\text{Green has opportunity}\} &= 2/6\\
\Pr\{\text{Scarlet has opportunity}\} &= 3/6
\end{aligned}$$

Exactly one has the *revolver* with the following probabilities, regardless of who has an opportuntity:

$$\begin{aligned}
\Pr\{\text{Plum has revolver}\} &= 4/8\\
\Pr\{\text{Green has revolver}\} &= 3/8\\
\Pr\{\text{Scarlet has revolver}\} &= 1/8
\end{aligned}$$

(a) Draw a tree diagram for this problem. Indicate edge and outcome probabilities.

Solution.



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(b) What is the probability that Colonel Mustard is shot?

Solution. Denote each outcome with a pair indicating who has the opportunity and who has the revolver. In this notation, the event that Colonel Mustard is shot consists of all outcomes where a single person has both:

$$\{(P, P), (G, G), (S, S)\}$$

The probability of this event is the sum of the outcome probabilities:

$$\begin{aligned}
\Pr\{\{(P, P), (G, G), (S, S)\}\} &= \Pr\{(P, P)\} + \Pr\{(G, G)\} + \Pr\{(S, S)\} \\
&= 4/48 + 6/48 + 3/48 \\
&= 13/48
\end{aligned}$$

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(c) What is the probability that Colonel Mustard is shot, given that Miss Scarlet does not have the revolver?

Solution. Let S be the event that Colonel Mustard is shot, and let N be the event that

Miss Scarlet does *not* have the revolver. The solution is:

$$\begin{aligned}
\Pr\{S \mid N\} &= \frac{\Pr\{S \cap N\}}{\Pr\{N\}} \\
&= \frac{\Pr\{(P,P),(G,G)\}}{\Pr\{(P,P),(P,G),(G,P),(G,G),(S,P),(S,G)\}} \\
&= \frac{\frac{4}{48}+\frac{6}{48}}{\frac{4}{48}+\frac{3}{48}+\frac{8}{48}+\frac{6}{48}+\frac{12}{48}+\frac{9}{48}} \\
&= \frac{5}{21}
\end{aligned}$$

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(d) What is the probability that Mr. Green had an opportunity, given that Colonel Mustard was shot?

Solution. Let G be the event that Mr. Green has an opportunity, and again let S be the event that Colonel Mustard is shot. Then the solution is:

$$\begin{aligned}
\Pr\{G \mid S\} &= \frac{\Pr\{G \cap S\}}{\Pr\{S\}} \\
&= \frac{\Pr\{(G,G)\}}{\Pr\{(P,P),(G,G),(S,S)\}} \\
&= \frac{\frac{6}{48}}{\frac{4}{48}+\frac{6}{48}+\frac{3}{48}} \\
&= \frac{6}{13}
\end{aligned}$$

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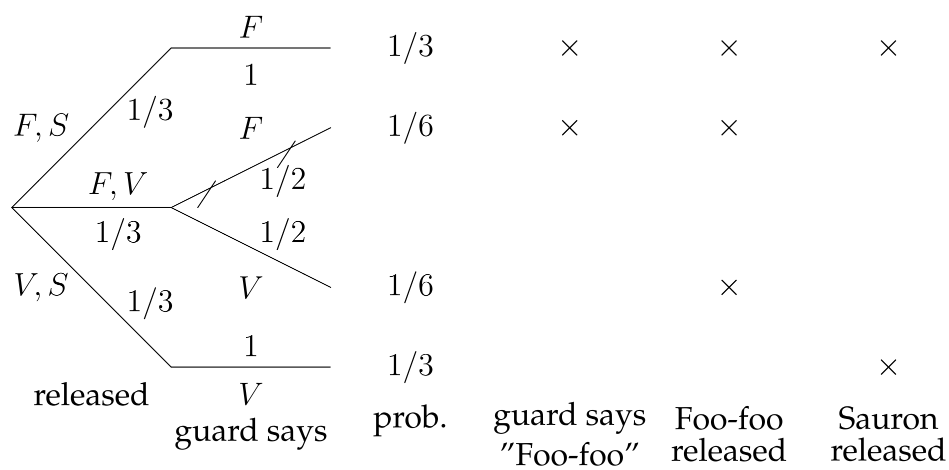
Problem 2. There are three prisoners in a maximum-security prison for fictional villains: the Evil Wizard Voldemort, the Dark Lord Sauron, and Little Bunny Foo-Foo. The parole board has declared that it will release two of the three, chosen uniformly at random, but has not yet released their names. Naturally, Sauron figures that he will be released to his home in Mordor, where the shadows lie, with probability $2/3$.

A guard offers to tell Sauron the name of one of the other prisoners who will be released (either Voldemort or Foo-Foo). However, Sauron declines this offer. He reasons that if the guard says, for example, "Little Bunny Foo-Foo will be released", then his own probability of release will drop to $1/2$. This is because he will then know that either he or Voldemort will also be released, and these two events are equally likely.

Using a tree diagram and the four-step method, either prove that the Dark Lord Sauron has reasoned correctly or prove that he is wrong. Assume that if the guard has a choice of naming either Voldemort or Foo-Foo (because both are to be released), then he names one of the two uniformly at random.

Solution. Sauron has reasoned incorrectly. In order to understand his error, let's begin by working out the sample space, noting events of interest, and computing outcome probabilities:



Define the events S, F, and "F" as follows:

$$\begin{aligned}
\text{“}F\text{”} &= \text{Guard says Foo-Foo is released} \\
F &= \text{Foo-Foo is released} \\
S &= \text{Sauron is released}
\end{aligned}$$

The outcomes in each of these events are noted in the tree diagram.

Sauron's error is in failing to realize that the event F (Foo-foo will be released) is different from the event "F" (the guard *says* Foo-foo will be released). In particular, the probability that Sauron is released, given that Foo-foo is released, is indeed $1/2$:

$$\begin{aligned}
\Pr\{S \mid F\} &= \frac{\Pr\{S \cap F\}}{\Pr\{F\}} \\
&= \frac{\frac{1}{3}}{\frac{1}{3} + \frac{1}{6} + \frac{1}{6}} \\
&= \frac{1}{2}
\end{aligned}$$

But the probability that Sauron is released given that the guard merely *says so* is still $2/3$:

$$\begin{aligned}\Pr\{S \mid \text{``}F\text{''}\} &= \frac{\Pr\{S \cap \text{``}F\text{''}\}}{\Pr\{\text{``}F\text{''}\}} \\ &= \frac{\frac{1}{3}}{\frac{1}{3}+\frac{1}{6}} \\ &= \frac{2}{3}\end{aligned}$$

So Sauron's probability of release is actually unchanged by the guard's statement. ■

Problem 3. You shuffle a deck of cards and deal your friend a 5-card hand.

(a) Suppose your friend says, "I have the ace of spades." What is the probablity that she has another ace?

Solution. The sample space for this experient is the set of all 5-card hands. All outcomes are equally likely, so the probability of each outcome is $1/\binom{52}{5}$. Let S be the event that your friend has the ace of spades, and let A be the event that your friend has another ace. Our objective is to compute:

$$\Pr\{A \mid S\} = \frac{\Pr\{A \cap S\}}{\Pr\{S\}}$$

The number of hands containing the ace of spades is equal to the number of ways to select 4 of the remaining 51 cards. Therefore:

$$\Pr\{S\} = \frac{\binom{51}{4}}{\binom{52}{5}}$$

The number of hands containing the ace of spades and at least one more ace is:

$$A \cap S = \binom{3}{1}\binom{48}{3} + \binom{3}{2}\binom{48}{2} + \binom{3}{3}\binom{48}{1}$$

Here the first term counts the number of hands with one additional ace, since there are $\binom{3}{1}$ ways to choose the extra ace and $\binom{48}{3}$ ways to choose the other cards. Similarly, the second term counts hands with two additional aces, and the last term counts hands with all three remaining aces. In probability terms, we have:

$$\Pr\{A \cap S\} = \frac{\binom{3}{1}\binom{48}{3} + \binom{3}{2}\binom{48}{2} + \binom{3}{3}\binom{48}{1}}{\binom{52}{5}}$$

Substituting these results into our original equation gives the solution:

$$\Pr\{A \mid S\} = \frac{\binom{3}{1}\binom{48}{3} + \binom{3}{2}\binom{48}{2} + \binom{3}{3}\binom{48}{1}}{\binom{51}{4}} = 0.2214\ldots$$

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(b) Suppose your friend says, "I have an ace." What is the probability that she has another ace?

Solution. The sample space and outcome probabilities are the same as before. Let L be the event that your friend has at least one ace, and M be the event that your friend has more than one ace. Our goal it to compute:

$$\Pr\{M \mid L\} = \frac{\Pr\{M \cap L\}}{\Pr\{L\}} = \frac{\Pr\{M\}}{\Pr\{L\}}$$

The second equality holds because your friend surely at least one ace if she has more than one; that is, $M \subseteq L$. The probability that your friend has at least one ace is:

$$\Pr\{L\} = \frac{\binom{4}{1}\binom{48}{4} + \binom{4}{2}\binom{48}{3} + \binom{4}{3}\binom{48}{2} + \binom{4}{4}\binom{48}{1}}{\binom{52}{5}}$$

The first term counts hands with a single ace, since there are $\binom{4}{1}$ ways to choose the ace and $\binom{48}{4}$ ways to choose the remaining four cards. The remaining terms are similar. The probability that your friend has more than one ace is:

$$\Pr\{M\} = \frac{\binom{4}{2}\binom{48}{3} + \binom{4}{3}\binom{48}{2} + \binom{4}{4}\binom{48}{1}}{\binom{52}{5}}$$

Plugging these results into the original equation gives:

$$\Pr\{M \mid L\} = \frac{\binom{4}{2}\binom{48}{3} + \binom{4}{3}\binom{48}{2} + \binom{4}{4}\binom{48}{1}}{\binom{4}{1}\binom{48}{4} + \binom{4}{2}\binom{48}{3} + \binom{4}{3}\binom{48}{2} + \binom{4}{4}\binom{48}{1}} = 0.12285\ldots$$

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(c) Are your answers to (a) and (b) the same? Explain why.

Solution. The answers are different. There are four aces, so there are sixteen different subsets of aces that your friend could have.

- If your friend says, "I have the ace of spades", then *eight* of these subsets are ruled out: those not containing the ace of spades.

- However, if your friend says, "I have an ace", then only *one* subset is ruled out: the subset containing no aces.

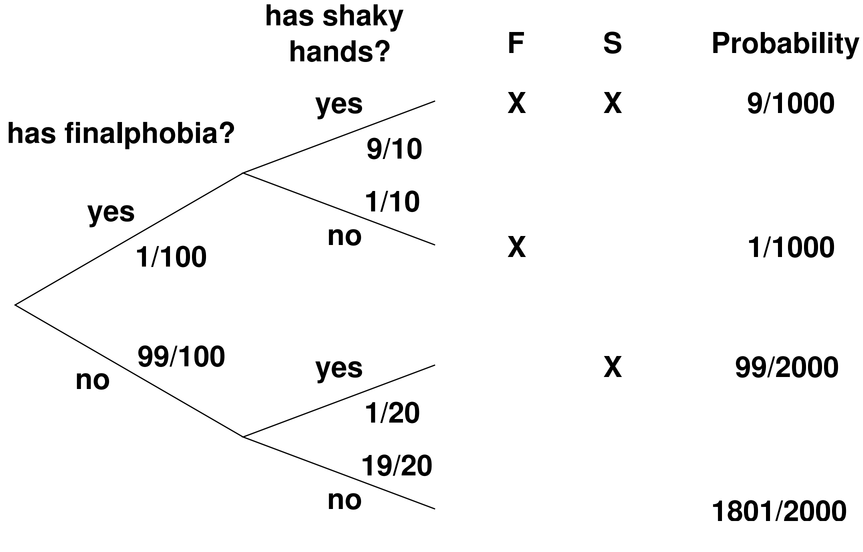
Thus, the probability that your friend has a second ace is different in these two cases, because we are conditioning on two very different events! ■

Problem 4. *Finalphobia* is a rare disease in which the victim has the delusion that he or she is being subjected to an intense mathematical examination.

- A person selected uniformly at random has finalphobia with probability $1/100$.
- A person with finalphobia has shaky hands with probability $9/10$.
- A person without finalphobia has shaky hands with probability $1/20$.

What is the probablility that a person selected uniformly at random has finalphobia, given that he or she has shaky hands?

Solution. Let F be the event that the randomly-selected person has finalphobia, and let S be the event that he or she has shaky hands. A tree diagram is worked out below:



The probability that a person has finalphobia, given that he or she has shaky hands is:

$$\begin{aligned}\Pr\{F \mid S\} &= \frac{\Pr\{F \cap S\}}{\Pr\{S\}} \\ &= \frac{9/1000}{9/1000 + 99/2000} \\ &= \frac{18}{18+99} \\ &= \frac{18}{117}\end{aligned}$$

So, while it's true that someone with shaky hands is five times more likely to have finalphobia than someone with steady hands, it remains a poor bet –about 1 in 5 –that someone with shaky hands actually has does have finalphobia. ■

Problem 5. Outside of their hum-drum duties as 6.042 TAs, Sayan is trying to learn to levitate using only intense concentration and Jelani is launching a "Nelson 2008" presidential campaign. Suppose that Sayan's probability of levitating is $1/6$, Jelani's chance of becoming president is $1/4$, and the success of one does not alter the other's chances.

(a) If at least one of them succeeds, what is the probability that Sayan learns to levitate?

Solution. Let L be the event that Sayan learns to levitate, and let P be the event that Jelani becomes president. We can work out the desired probability as follows:

$$\begin{aligned}\Pr\{L \mid (L \cup P)\} &= \frac{\Pr\{L \cap (L \cup P)\}}{\Pr\{L \cup P\}} \\ &= \frac{\Pr\{L\}}{1 - \Pr\left\{\overline{L} \cap \overline{P}\right\}} \\ &= \frac{1/6}{1-(1-1/6)(1-1/4)} \\ &= \frac{4}{9}\end{aligned}$$

The first step uses the definition of conditional probability. In the second step, we rewrite both the top and bottom of the fraction using set identities. Then we substitute in the given probability and simplify. ■

(b) If at most one of them succeeds, what is the probability that Jelani becomes the president of the United States?

Solution. Define events L and P as before.

$$\begin{aligned}
\Pr\left\{P \mid \left(\overline{L} \cup \overline{P}\right)\right\} &= \frac{\Pr\left\{P \cap \left(\overline{L} \cup \overline{P}\right)\right\}}{\Pr\left\{\overline{L} \cup \overline{P}\right\}} \\
&= \frac{\Pr\left\{P \cap \overline{L}\right\}}{1 - \Pr\{L \cap P\}} \\
&= \frac{(1/4) \cdot (5/6)}{1 - (1/6) \cdot (1/4)} \\
&= \frac{5}{23}
\end{aligned}$$

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(c) If exactly one of them succeeds, what is the probability that it is Sayan?

Solution.

$$\begin{aligned}
\Pr\left\{L \mid \left(\left(L \cap \overline{P}\right) \cup \left(\overline{L} \cap P\right)\right)\right\} &= \frac{\Pr\left\{L \cap \overline{P}\right\}}{\Pr\left\{(L \cap \overline{P}) \cup (\overline{L} \cap P)\right\}} \\
&= \frac{(1/6) \cdot (3/4)}{(1/6) \cdot (3/4) + (5/6) \cdot (1/4)} \\
&= \frac{3}{8}
\end{aligned}$$

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Problem 6. Suppose n balls are thrown randomly into n boxes, so each ball lands in each box with uniform probability. Also, suppose the outcome of each throw is independent of all the other throws.

(a) Let X_i be an indicator random variable whose value is 1 if box i is empty and 0 otherwise. Write a simple closed form expression for the probability distribution of X_i. Are $X_1, X_2, \ldots, X_n$ independent random variables?

Solution. Box i is empty iff all n balls land in other boxes. The probability that a ball will land in another box in $(n-1)/n = 1-(1/n)$, and since the balls are thrown independently, we have

$$\Pr\{X_i = 1\} = \left(1 - \frac{1}{n}\right)^n. \tag{1}$$

The X_i's are not independent. For example,

$$\Pr\{X_1 = X_2 = \cdots = X_n = 1\} = 0 < \prod_{i=1}^{n} \Pr\{X_i = 1\}.$$

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(b) Show that

$$\Pr\{\text{at least } k \text{ balls fall in the first box}\} \leq \binom{n}{k}\left(\frac{1}{n}\right)^k.$$

Solution. Let S be a set of k of the n balls, and let E_S be the event that each of these k balls falls in the first box. Since the probability that a ball lands in this box is $1/n$, and the throws are independent, we have

$$\Pr\{E_S\} = \left(\frac{1}{n}\right)^k. \tag{2}$$

The event that *at least* k balls land in the first box is the union of all the events E_S. There are $\binom{n}{k}$ subsets, S, of k balls, so by the Union Bound,

$$\Pr\{\text{at least } k \text{ balls fall in the first box}\} \leq \binom{n}{k} \cdot \Pr\{E_S\}.$$

Using the value for $\Pr\{E_S\}$ from (**??**) in the preceding inequality yields the required bound. ■

(c) Let R be the maximum of the numbers of balls that land in each of the boxes. Conclude from the previous parts that

$$\Pr\{R \geq k\} \leq \frac{n}{k!}.$$

Solution. Note that $R \geq k$ exactly when some box has at least k balls. Since the bound on the probability of at least k balls in the first box applies just as well to any box, we can apply the Union Bound to having at least k balls in at least one of the n boxes:

$$\Pr\{R \geq k\} \leq n \Pr\{\text{at least } k \text{ balls fall in the first box}\}.$$

So from the previous problem part, we have

$$\begin{aligned}
\Pr\{R \geq k\} &\leq n\binom{n}{k}\left(\frac{1}{n}\right)^k \\
&= n\left(\frac{n(n-1)\cdots(n-k+1)}{k!\,n^k}\right) \\
&= \frac{n}{k!}\left(\frac{n}{n}\cdot\frac{n-1}{n}\cdots\frac{n-k+1}{n}\right) \\
&\leq \frac{n}{k!}
\end{aligned}$$

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(d) Conclude that

$$\lim_{n\to\infty} \Pr\{R \geq n^\epsilon\} = 0$$

for all $\epsilon > 0$.

Solution. Using Stirling's formula, and the upper bound from the previous part, we have

$$\Pr\{R \geq k\} \leq \frac{n}{k!} \sim \frac{n}{\sqrt{2\pi k}(k/e)^k} \leq \frac{n}{(k/e)^k} = \frac{ne^k}{k^k} = \frac{e^{k+\ln n}}{e^{k\ln k}}.$$

Now let $k = n^\epsilon$. Then the exponent of e in the numerator above is $n^\epsilon + \ln n$, and the exponent of e in the denominator is $n^\epsilon \ln n^\epsilon = \epsilon n^\epsilon \ln n$. Since

$$n^\epsilon + \ln n = o(n^\epsilon \ln n^\epsilon),$$

we conclude

$$\Pr\{R \geq n^\epsilon\} \leq \frac{e^{n^\epsilon+\ln n}}{e^{\epsilon n^\epsilon \ln n}} \to 0$$

as n approaches ∞. ■

Problem 7. (An open-ended discussion question.) Consider a set, S, consisting of 77 twenty-one digit numbers. We can use the pigeonhole principle to prove that two distinct subsets of the numbers in S have the same sum, but actually finding two such sets is can be difficult. Naively, we could sum the elements in all 2^{77} subsets and find two that match, but this is a huge computational task.

Recall the birthday principle: If there are d days in a year and $\sqrt{2d}$ people in a room, then the probability that two share a birthday is about $1 - 1/e = 0.632\ldots$.

How could the birthday principle help you find two distinct subsets of S with the same sum using significantly fewer than 2^{77} operations —say only a trillion operations? What assumptions must you make?

Solution. Assume that the sums are uniformly distributed, mutually independent random variables taking on values in the range $[0, \ldots, 77 \cdot 10^{21}]$. Then we have a good chance of finding two subsets with the same sum if the number of subsets we consider is about:

$$\sqrt{2 \cdot 77 \cdot 10^{21}} \approx 4 \cdot 10^{11}$$

Of course, the sums are not uniformly distributed; in fact, the sum of a random subset is likely to be close to the expected value. This nonuniformity only improves the computational picture, however. (If almost everyone were born in July, then finding two people with the same birthday would be easier.)

The sums are also not mutually independent. However, if we sum a few hundred billion selected at random, then the sets are likely to be different enough that their sums are "mostly kinda" independent.

Overall, it seems that several billion sums should suffice to find two subsets with the same sum. ■