Problem Set 3

Due: October 3

Reading: Week 4 Notes.

Problem 1. **(a)** List all the different binary relations on the set $\{0, 1\}$.

(b) Over the domain $\{0, 1\}$, which of these relations are weak partial orders? strict partial orders? equivalence relations?

Problem 2. We partially order the power set, $\mathcal{P}(\{1, 2, \ldots, n\})$, by the subset relation, $\subseteq$.

(a) Describe a maximum length chain in $\mathcal{P}(\{1, 2, \ldots, n\})$. Briefly explain why there can't be a longer chain than the one you described.

(b) Describe a topological sort of $\mathcal{P}(\{1, 2, \ldots, n\})$, with a brief justification that your sort is correct.

(c) Use Dilworth's Lemma to show that there must be an antichain of size $\geq 2^n/(n+1)$. Describe the biggest antichain that you can find.

Problem 3. Consider the natural numbers partially ordered by divisibility.

(a) Prove that this partial order has an infinite chain.

(b) Prove that this partial order has an infinite antichain.

(c) Now restrict the domain to the natural numbers $\leq n$. Consider the chain $1 \preceq_R 2 \preceq_R 4 \preceq_R \ldots \preceq_R 2^{\lfloor \log_2 n \rfloor}$. Prove that it is maximal.

(d) Let c be the length of the power of 2 chain. By Dilworth's Lemma there is an antichain of length n/c. Describe one.

Problem 4. We consider DAG's where each vertex represents a task to be completed. If there is a path from one vertex, v, to another vertex, w, then the v task must be completed before the w task. Assuming all tasks take unit time to complete, we showed in the Notes that the minimum time schedule to complete all the tasks is the size (number of vertices), t, of the longest path (chain) in the DAG.

Formally, a *schedule* for a DAG is a partition of the vertices. Each block of the partition is supposed to correspond to a set of tasks that are to be performed simultaneously. The *number of processors required* by a schedule is the maximum number of tasks that are scheduled to be performed simultaneously.

(a) Describe purely in terms of graph, partition, and partial order properties (no informal descriptions in terms of "jobs," "parallel processing," etc.):

- exactly the properties a vertex partition of a DAG must satisfy in order to represent a possible schedule for the vertex tasks,

- the total time required to complete a schedule,

- the number of processors required by a schedule.

(b) Give a small example of a DAG with more than one minimum time schedule.

(c) Explain why any schedule that requires only p processors to complete n tasks must take time at least $\lceil n/p \rceil$.

(d) Let $D_{n,t}$ be the DAG with n vertices that consists of a directed path of $t - 1$ vertices ending with edges from the final, $(t - 1)$st, vertex on the path directly to each of the remaining $n - (t - 1)$ vertices, as in the following figure:
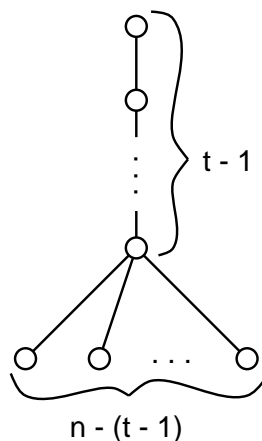
What is the minimum time schedule for $D_{n,t}$? Explain why it is unique. How many processors does it require?

(e) Describe a minimum time p-processor schedule for $D_{n,t}$. Write a simple formula for this minimum time, $M(n, t, p)$.

(f) Show that *every* DAG with n vertices and maximum chain size, t, has a p-processor schedule that runs in time $M(n, t, p)$.

Hint: Induction – you decide on what variable. You may find it helpful to use the fact that if $a \geq b \geq 0$, then

$$\lceil a - b \rceil \leq 1 + \lceil a \rceil - \lceil b \rceil \tag{1}$$

for all real numbers a, b.

Massachusetts Institute of Technology
6.042J/18.062J, Fall '05: Mathematics for Computer Science
Prof. Albert R. Meyer and Prof. Ronitt Rubinfeld

Solutions cover sheet
September 26

Student's Solutions to Problem Set 3

Your name:

Due date: October 3

Submission date:

Circle your TA: David Jelani Sayan Hanson

Collaboration statement: Circle one of the two choices and provide all pertinent info.

1. I worked alone and only with course materials.

2. I collaborated on this assignment with:

   got help from:[1]

   and referred to:[2]

DO NOT WRITE BELOW THIS LINE

Problem	Score
1	
2	
3	
4	
Total	

[1]People other than course staff.
[2]Give citations to texts and material other than the Fall '02 course materials.