

**Recitation 20 Solutions: November 18, 2010**

1. (a) Let  $X_i$  be a random variable indicating the quality of the  $i$ th bulb (“1” for good bulbs, “0” for bad ones).  $X_i$ ’s are independent Bernoulli random variables. Let  $Z_n$  be

$$Z_n = \frac{X_1 + X_2 + \dots + X_n}{n}.$$
$$\mathbf{E}[Z_n] = p \quad \text{var}(Z_n) = \frac{n \text{var}(X_i)}{n^2} = \frac{\sigma^2}{n},$$

where  $\sigma^2$  is the variance of  $X_i$ .

Applying Chebyshev’s inequality yields,

$$\mathbf{P}(|Z_n - p| \geq \epsilon) \leq \frac{\sigma^2}{n\epsilon^2},$$

As  $n \rightarrow \infty$ ,  $\frac{\sigma^2}{n\epsilon^2} \rightarrow 0$  and  $\mathbf{P}(|Z_n - p| \geq \epsilon) \rightarrow 0$ .  
Hence,  $Z_n$  converges to  $p$  in probability.

- (b) By Chebyshev’s inequality,

$$\mathbf{P}(|Z_{50} - p| \geq 0.1) \leq \frac{\sigma^2}{50(0.1)^2},$$

Since  $X_i$  is a Bernoulli random variable, its variance  $\sigma^2$  is  $p(1-p)$ , which is less than or equal to  $\frac{1}{4}$ . Thus,

$$\mathbf{P}(|Z_{50} - p| \geq 0.1) \leq \frac{1/4}{50(0.1)^2} = 0.5$$

- (c) By Chebyshev’s inequality,

$$\mathbf{P}(|Z_n - p| \geq 0.1) \leq \frac{\sigma^2}{n\epsilon^2} \leq \frac{1/4}{n(0.1)^2}$$

To guarantee a probability 0.95 of falling in the desired range,

$$\frac{1/4}{n(0.1)^2} < 0.05,$$

which yields  $n \geq 500$ . Note that  $n \geq 500$  guarantees the accuracy specification even for the highest variance, namely  $1/4$ . For smaller variances, we need smaller values of  $n$  to guarantee the desired accuracy. For example, if  $\sigma^2 = 1/16$ ,  $n \geq 125$  would suffice.

2. (a)  $\mathbf{E}[X_n] = 0 \cdot \left(1 - \frac{1}{n}\right) + 1 \cdot \frac{1}{n} = \frac{1}{n}$   
 $\text{var}(X_n) = \left(0 - \frac{1}{n}\right)^2 \cdot \left(1 - \frac{1}{n}\right) + \left(1 - \frac{1}{n}\right)^2 \cdot \left(\frac{1}{n}\right) = \frac{n-1}{n^2}$   
 $\mathbf{E}[Y_n] = 0 \cdot \left(1 - \frac{1}{n}\right) + n \cdot \frac{1}{n} = 1$   
 $\text{var}(Y_n) = (0-1)^2 \cdot \left(1 - \frac{1}{n}\right) + (n-1)^2 \cdot \left(\frac{1}{n}\right) = n-1$

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(b) Using Chebyshev's inequality, we have

$$\lim_{n \rightarrow \infty} \mathbf{P}(|X_n - \frac{1}{n}| \geq \epsilon) \leq \lim_{n \rightarrow \infty} \frac{n-1}{n^2 \epsilon^2} = 0$$

Moreover,  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$ .

It follows that  $X_n$  converges to 0 in probability. For  $Y_n$ , Chebyshev suggests that,

$$\lim_{n \rightarrow \infty} \mathbf{P}(|Y_n - 1| \geq \epsilon) \leq \lim_{n \rightarrow \infty} \frac{n-1}{\epsilon^2} = \infty,$$

Thus, we cannot conclude anything about the convergence of  $Y_n$  through Chebyshev's inequality.

(c) For every  $\epsilon > 0$ ,

$$\lim_{n \rightarrow \infty} \mathbf{P}(|Y_n| \geq \epsilon) \leq \lim_{n \rightarrow \infty} \frac{1}{n} = 0,$$

Thus,  $Y_n$  converges to zero in probability.

(d) The statement is false. A counter example is  $Y_n$ . It converges in probability to 0 yet its expected value is 1 for all  $n$ .

(e) Using the Markov bound, we have

$$\mathbf{P}(|X_n - c| \geq \epsilon) = P(|X_n - c|^2 \geq \epsilon^2) \leq \frac{\mathbf{E}[(X_n - c)^2]}{\epsilon^2}.$$

Taking the limit as  $n \rightarrow \infty$ , we obtain

$$\lim_{n \rightarrow \infty} \mathbf{P}(|X_n - c| \geq \epsilon) = 0,$$

which establishes convergence in probability.

(f) A counter example is  $Y_n$ .  $Y_n$  converges to 0 in probability, but

$$\mathbf{E}[(Y_n - 0)^2] = 0 \cdot \left(1 - \frac{1}{n}\right) + (n^2) \cdot \frac{1}{n} = n$$

Thus,

$$\lim_{n \rightarrow \infty} \mathbf{E}[(Y_n - 0)^2] = \infty,$$

and  $Y_n$  does not converge to 0 in the mean square.

3. (a) No. Since  $X_i$  for any  $i \geq 1$  is uniformly distributed between -1.0 and 1.0.

(b) Yes, to 0. Since for  $\epsilon > 0$ ,

$$\begin{aligned} \lim_{i \rightarrow \infty} \mathbf{P}(|Y_i - 0| > \epsilon) &= \lim_{i \rightarrow \infty} \mathbf{P}\left(\left|\frac{X_i}{i} - 0\right| > \epsilon\right) \\ &= \lim_{i \rightarrow \infty} [\mathbf{P}(X_i > i\epsilon) + \mathbf{P}(X_i < -i\epsilon)] = 0. \end{aligned}$$

(c) Yes, to 0. Since for  $\epsilon > 0$ ,

$$\begin{aligned} \lim_{i \rightarrow \infty} \mathbf{P}(|Z_i - 0| > \epsilon) &= \lim_{i \rightarrow \infty} \mathbf{P}(|(X_i)^i - 0| > \epsilon) \\ &= \lim_{i \rightarrow \infty} \left[\mathbf{P}(X_i > \epsilon^{\frac{1}{i}}) + \mathbf{P}(X_i < -(\epsilon^{\frac{1}{i}}))\right] \\ &= \lim_{i \rightarrow \infty} \left[\frac{1}{2}(1 - \epsilon^{\frac{1}{i}}) + \frac{1}{2}(1 - \epsilon^{\frac{1}{i}})\right] = \lim_{i \rightarrow \infty} (1 - \sqrt[i]{\epsilon}) \\ &= 0. \end{aligned}$$

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