

6.041SC Probabilistic Systems Analysis and Applied Probability, Fall 2013 Transcript – Tutorial: Mean First Passage and Recurrence Times

In this problem, we are looking at a student whose performance from day to day sort of oscillates according to a Markov chain. In particular, the student can either be in state 1, which is a state of being up to date, or in state 2, which is a state of being kind of fallen behind. Now, the transition probabilities between these two states are given by the numbers here, which is 0.2 from state 1 to 2, 0.6 from 2 to 1, 0.4 from 2 back to 2, and 0.8 from 1 back to state 1.

The quantity we're interesting calculating is this notion of first passage time. Let me define what that means. Suppose we are looking at a time horizon of time 0, 1, 2, 3. And let's call the state of the Markov chain x of t . Suppose we start from the chain being in state 2 here.

Now, if we look at a particular sample path, let's say 2 and 2 again on day 1, and 2 again on day 2, and on day 3, the student enters state 1. So in this sample path, we start from time 0 and time 3 is the first time we enter state 1. And we'll say that the first passage time, namely, the first time we enter state 1 in this case, is equal to 3. More formally, we'll define t_j as the first pass the time to state 1 conditional on that we start from state j at time 0.

Now, this quantity, of course, is random. Depending on the realization, we have different numbers. And we are interested in calculating the expected value of t_2 . That is, on average, if we start from state 2 here, how long would it take for us to enter state 1?

Now to calculate this quantity, in the following recursion will be very important. The idea is we don't know exactly what t_2 is. But t_2 has to satisfy a certain recurrent equation, namely, t_2 must be equal to 1 plus summation j equal to 1 to 2 $P_{2j}t_j$.

Now let me explain what this equation means. Let's say we are at state 2. Well, we don't actually know how long it's going to take for us to enter state 1. But we do know after one step, I will be go into some other state. Let's call it state j . And from state j , it's going to take some time to enter state 1 finally. So this equation essentially says the time for us to first enter state 1 from 2 is 1-- which is the next step-- plus the expected time from that point on to enter 1. So that constitutes our [? recurrent ?] relationship.

Now, by this definition, we can see that this is simply 1 plus P_{21} times t_1 plus P_{22} times t_2 . Now, the definition of t_j says t_1 must be 0 because, by definition, if we start from state 1, we are already in state 1. So the time to reach state 1 is simply 0. So this term disappears. And we end up with 1 plus $P_{22} t_2$. If we plug in a number of P_{22} -- which is 0.4 right here-- we get 1 plus 0.4 t_2 .

Now we started from t_2 and we ended up with another expression involving numbers and only one unknown, which is t_2 . Combining this together and solving for t_2 , we get t_2 equals 1 divided by 1 minus 0.4, which is $5/3$. And that is the answer for the first part of the problem.

In the second part of the problem, we are asked to do something similar as before but with a slight twist. Here, I copied over the definition for t_j , which is the first time to visit state 1 starting from state j at time t equals 0. And the little t_j is this expectation.

And here we're going to define a similar quantity, which is t_1^* , let's say, star, defined as the first time to visit state 1 again. So that's the recurrence part starting from state 1, 1 at t equals 0. So this is the recurrence time from state 1 back to state 1 again.

As an example, again, we look at t equals 0, 1, 2, 3, 4. And here, if we start from state 1 on time 0, we went to state 2, 2, 1, 1 again. Now here, again, time 3 will be the first time to visit state 1 after time 0. And we don't count the very first 0. And that will be our t_1^* . So t_1^* in this particular case is equal to 3. OK.

Same as before, we like to calculate the expected time to revisit state 1. Define little t_1^* expected value of t_1^* . And we'll be using the same recurrence trick through the following equation. We say that t_1^* is equal to $1 + \sum_{j \neq 1} P_{1j} t_j$.

Now, since we started from state 1, this goes from 1 to state $1j$ and t_j . Again, the interpretation is we started at state 1 at time t equals 0, we went to some other state-- we call it j -- and from state j , it goes around, and after time expected value t_j , we came back to state 1. Here, and as before, this equation works because we are working with a Markov chain whereby the time to reach some other state only depends on the current state. And that's why we're able to break down the recursion as follows.

If we write out the recursion, we get $1 + P_{11} t_1 + P_{12} t_2$. As before, t_1 now is just the expected first passage time from state 1. And by definition, it is 0. Because if we start from state 1, it's already in state 1 and takes 0 time to get there. So again, like before, this term goes out. And we have $1 + 0.2 \times \frac{5}{3}$. And this number came from the previous calculation of t_2 . And this gives us $\frac{4}{3}$.

So this completes the problem. And just to remind ourselves, the kind of crux of the problem is this type of recursion which expresses a certain quantity in the one incremental step followed by the expected time to reach a certain destination after that one step. And we can do so because the dynamics is modeled by a Markov chain. And hence, the time to reach a certain destination after this first step only depends on where you start again, in this case, state j .

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