Recitation 7 Solutions
September 30, 2010

1. See the textbook, Problem 2.35, page 130.

2. (a)

$$\begin{aligned} p_X(1) &= \mathbf{P}(X=1,Y=1)+\mathbf{P}(X=1,Y=2)+\mathbf{P}(X=1,Y=3) \\ &= 1/12+2/12+1/12 = 1/3 \end{aligned}$$

   (b) The solution is a sketch of the following conditional PMF:

$$p_{Y|X}(y \mid 1) = \frac{p_{Y,X}(y,1)}{p_X(1)} = \begin{cases} 1/4, & \text{if } y=1, \\ 1/2, & \text{if } y=2, \\ 1/4, & \text{if } y=3, \\ 0, & \text{otherwise.} \end{cases}$$

   (c) $\mathbf{E}[Y \mid X=1] = \sum_{y=1}^{3} y\, p_{Y|X}(y \mid 1) = 1\cdot\frac{1}{4} + 2\cdot\frac{1}{2} + 3\cdot\frac{1}{4} = 2$

   (d) Assume that X and Y are independent. Because $p_{X,Y}(3,1)=0$ and $p_Y(1)=1/4$, $p_X(3)$ must equal zero. This further implies $p_{X,Y}(3,2)=0$ and $p_{X,Y}(3,3)=0$. All the remaining probability mass must go to $(X,Y)=(2,2)$, making $p_{X,Y}(2,2)=5/12$, $p_X(2)=8/12$, and $p_Y(2)=7/12$. However, $p_{X,Y}(2,2) \neq p_X(2)\cdot p_Y(2)$, contradicting the assumption; thus X and Y are not independent.

   A simpler explanation uses only two X values and two Y values for which all four (X,Y) pairs have specified probabilities. Note that if X and Y are independent, then $p_{X,Y}(1,3)/p_{X,Y}(1,1)$ and $p_{X,Y}(2,3)/p_{X,Y}(2,1)$ must be equal because they must both equal $p_Y(3)/p_Y(1)$. This necessary equality does not hold, so X and Y are not independent.

   (e) Knowing that X and Y are conditionally independent given B, we must have

$$\frac{p_{X,Y}(1,1)}{p_{X,Y}(1,2)} = \frac{p_{X,Y}(2,1)}{p_{X,Y}(2,2)}$$

   since the (X,Y) pairs in the equality are all in B. Thus

$$p_{X,Y}(2,2) = \frac{p_{X,Y}(1,2)p_{X,Y}(2,1)}{p_{X,Y}(1,1)} = \frac{(2/12)(2/12)}{1/12} = \frac{4}{12} = \frac{1}{3}.$$

   (f) Since $\mathbf{P}(B) = 9/12 = 3/4$, we normalize to obtain $p_{X,Y|B}(2,2) = \frac{p_{X,Y}(2,2)}{\mathbf{P}(B)} = 4/9$.

3. See the textbook, Problem 2.33, page 128.