6.041SC Probabilistic Systems Analysis and Applied Probability, Fall 2013 Transcript – Recitation: A Derived Distribution Example

Hi. In this problem we'll work through an example of calculating a distribution for a minute variable using the method of derived distributions. So in general, the process goes as follows. We know the distribution for some random variable X and what we want is the distribution for another random variable of Y, which is somehow related to X through some function g. So Y is a g of X.

And the steps that we follow-- we can actually just kind of summarize them using this four steps. The first step is to write out the CDF of Y. So Y is thing that we want. And what we'll do is we'll write out the CDF first. So remember the CDF is just capital F of y, y is the probability that random variable Y is less than or equal to some value, little y.

The next thing we'll do is, we'll use this relationship that we know, between Y and X. And we'll substitute in, instead of writing the random variable Y In here, we'll write it in terms of X. So we'll plug in for-- instead of Y, we'll plug-in X. And we'll use this function g in order to do that.

So what we have now is that up to here, we would have that the CDF of Y is now the probability that the random variable X is less than or equal to some value, little y. Next what we'll do is we'll actually rewrite this probability as a CDF of X. So the CDF of X, remember, would be-- F of x is that the probability of X is less than or equal to some little x.

And then once we have that, if we differentiate this-- when we differentiate the CDF of X, we get the PDF of X. And what we presume is that we know this PDF already. And from that, what we get is, when we differentiate this thing, we get the PDF of Y. So through this whole process what we get is, we'll get the relationship between the PDF of Y and the PDF of X. So that is the process for calculating the PDF of Y using X.

So let's go into our specific example. In this case, what we're told is that X, the one that we know, is a standard normal random variable. Meaning that it's mean 0 and variance 1. And so we know the form of the PDF. The PDF of x is this, 1 over square root of 2 pi e to the minus x squared over 2.

And then the next thing that we're told is this relationship between X and Y. So what we're told is, if X is negative, then Y is minus X. If X is positive, then Y is the square root of X. So this is a graphical its representation of the relationship between X and Y.

All right, so we have everything that we need. And now let's just go through this process and calculate what the PDF of Y is. So the first thing we do is we write out the PDF of Y. So the PDF of Y is what we've written. It's the probability that the random variable Y is less than or equal to some little y.

Now the next step that we do is we have to substitute in, instead of in terms of Y, we want to substitute it in terms of X. Because we actually know stuff about X, but we don't know anything about Y. So what is the probability that Y, the random variable Y, is less than or equal to some little y?

Well, let's go back to this relationship and see if we can figure that out. So let's pretend that here is our little y. Well, if the random variable Y is less than or equal to little y, it has to be underneath this horizontal line. And in order for it to be underneath this horizontal line, that means that X has to be between this range. And what is this range? This range goes from minus Y to Y squared.

So why is that? It's because in this portion X and Y are related as, Y is negative X and here it's Y is square root of X. So if X is Y squared, then Y would be Y. If X is negative Y, then Y would be Y. All right, so this is the range that we're looking for.

So if Y, the random variable Y is less than or equal to little y, then this is the same as if the random variable X is between negative Y and Y squared. So let's plug that in. This is the same as the probability that X is between negative Y and Y squared.

So those are the first two steps. Now the third step is, we have to rewrite this as the CDF of x. So right now we have it in terms of a probability of some event related to X. Let's actually transform that to be explicitly in terms of the CDF of X. So how do we do that?

Well, this is just the probability that X is within some range. So we can turn that into the CDF by writing it as a difference of two CDFs. So this is the same as the probability that X is less than or equal to Y squared minus the probability that X is less than or equal to negative Y.

So in order to find the probability that X is between this range, we take the probability that it's less than Y squared, which is everything here. And then we subtract that probability that it's less than Y, negative Y. So what we're left with is just within this range.

So these actually are now exactly CDFs of X. So this is F of X evaluated at Y squared and this is F of X evaluated at negative Y. So now we've completed step three. And the last step that we need to do is differentiate.

So if we differentiate both sides of this equation with respect to Y, we'll get that the left side would get what we want, which is the PDF of Y. Now we differentiate the right side-- we'll have to invoke the chain rule. So the first thing that we do is, well, this is a CDF of X. So when we differentiate we'll get the PDF of X.

But then we also have invoke the chain rule for this argument inside. So the derivative of Y squared would give us an extra term, 2Y. And then similarly this would give us the PDF of X evaluated at negative Y plus the chain will give us an extra term of negative 1. So let's just clean this up a little bit.

So it's 2y F X squared plus F X minus Y. All right, so now we're almost done. We've differentiated. We have the PDF of Y, which is what we're looking for. And we've written it in terms of the PDF of X. And fortunately we know what that is, so once we plug that in, then we're essentially done.

So what is the PDF? Well, the PDF of X evaluated at Y squared is going to give us 1 over square root of 2 pi e to the minus-- so in this case, X is Y squared-- so we get Y to the fourth over 2. And then we get another 1 over square root of 2 pi e to the minus Y squared over 2. OK, and now we're almost done.

The last thing that we need to take care of is, what is the range? Now remember, it's important when you calculate out PDFs to always think about the ranges where things are valid. So when we think about this, what is the range where this actually is valid? Well, Y, remember is related to X in this relationship. So as we look at this, we see that Y can never be negative.

Because no matter what X is, Y gets transformed into some non-negative version. So what we know is that this is now actually valid only for Y greater than 0 and for Y less than 0, the PDF is 0. So this gives us the final PDF of Y.

All right, so it seems like at first when you start doing these derived restriction problems that it's pretty difficult. But if we just remember that there are these pretty straightforward steps that we follow, and as long as you go through these steps and do them methodically, then you can actually come up with the solution for any of these problems. And one last thing to remember is to always think about what are the ranges where these things are valid? Because the relationship between these two random variables could be pretty complicated and you need to always be aware of when things are non-zero and when they are 0.

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