LECTURE 12

- Readings: Section 4.1

Lecture outline

- Definition of Transforms
- Why transforms?
- Moment Generating Property
- Examples
- Application to Sums of Indep. R.V.s

Definition of Transforms

$$M_X(s) = \mathbf{E}[e^{sX}]$$

- Discrete, PMF:

$$M_X(s) = \mathbf{E}[e^{sX}] = \sum_x e^{sx} p_X(x)$$

- Continuous, PDF:

$$M_X(s) = \mathbf{E}[e^{sX}] = \int_{-\infty}^{\infty} e^{sx} f_X(x)\,dx$$

- Inversion theorem:

Knowing the PMF or PDF	$\Longleftrightarrow$	Knowing the Transform

Why Transforms?

- A new kind of representation.

- Sometimes convenient for:
  - Calculations
  - Analytic Derivations
  - Theorem Proving

Moment Generating Property

- Moments: $\mathbf{E}[X^n]$, we need to integrate.
- Can instead differentiate the transform:

$$M_X(s) = \mathbf{E}[e^{sX}] = \begin{cases} \sum_x e^{sx} p_X(x) \\ \int_{-\infty}^{\infty} e^{sx} f_X(x)\,dx \end{cases}$$

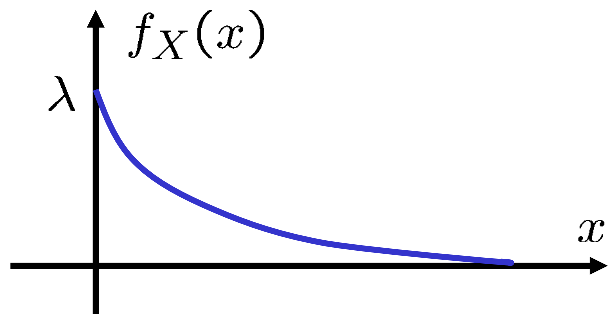
$$M_X(s)\big|_{s=0} = \mathbf{E}[e^{0X}] = 1$$

$$\frac{d}{ds} M_X(s)\bigg|_{s=0} = \mathbf{E}[X]$$

$$\boxed{\frac{d^n}{ds^n} M_X(s)\bigg|_{s=0} = \mathbf{E}[X^n]}$$

Example: **Exponential PDF**

$$f_X(x) = \lambda e^{-\lambda x} \text{ over } x \geq 0 \ (\lambda > 0)$$



$$M_X(s) = \lambda \int_0^\infty e^{sx} e^{-\lambda x} dx = \lambda \int_0^\infty e^{(s-\lambda)x} dx = \frac{\lambda}{\lambda - s}$$

$$\mathbf{E}[X] = \frac{d}{ds} M_X(s)\Big|_{s=0} = \frac{\lambda}{(\lambda - s)^2}\Big|_{s=0} = \frac{1}{\lambda}$$

- We can get all higher order moments, ($\mathbf{E}[X^2]$, etc.) in a similar fashion.

Example: **Geometric PMF**

- If we know X takes nonnegative integer values:

$$M_X(s) = \mathbf{E}[e^{sX}] = \sum_x e^{sx} p_X(x)$$
$$= p_X(0) + p_X(1)e^s + p_X(2)e^{2s} + \cdots$$

- Now, say we have: $M_X(s) = \dfrac{pe^s}{1-(1-p)e^s}$
- Recall: $\dfrac{1}{1-\alpha} = 1 + \alpha + \alpha^2 + \cdots$ for $|\alpha| < 1$

  So: $M_X(s) = pe^s\left(1 + (1-p)e^s + (1-p)^2 e^{2s} + \cdots\right)$

- We recognize: $p_X(x) = p(1-p)^{x-1}$ for $x = 1, 2, \cdots$

  This is the geometric PMF.

The Transform of $X + Y$.

- Let X, Y be two **independent** r.v.s.

- Let $W = X + Y$.

We convolve PMFs or PDFs	$\Longleftrightarrow$	We multiply Transforms

- We get: $M_W(s) = M_X(x)M_Y(s)$

Transform of the Normal PDF

- General normal X: $$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$
- Transform: $$M_X(s) = e^{(s^2\sigma^2/2)+s\mu}$$
- **Sum of independent normals**:

$$X \sim N(\mu_x, \sigma_x^2) \qquad Y \sim N(\mu_y, \sigma_y^2) \qquad W = X + Y$$

$$\begin{aligned} M_W(s) &= M_X(x) M_Y(s) \\ &= e^{(s^2\sigma_x^2/2)+s\mu_x} \cdot e^{(s^2\sigma_y^2/2)+s\mu_y} \\ &= e^{[s^2(\sigma_x^2+\sigma_y^2)/2+s(\mu_x+\mu_y)]} \end{aligned}$$

- Conclude: $W \sim N(\mu_x + \mu_y, \sigma_x^2 + \sigma_y^2)$