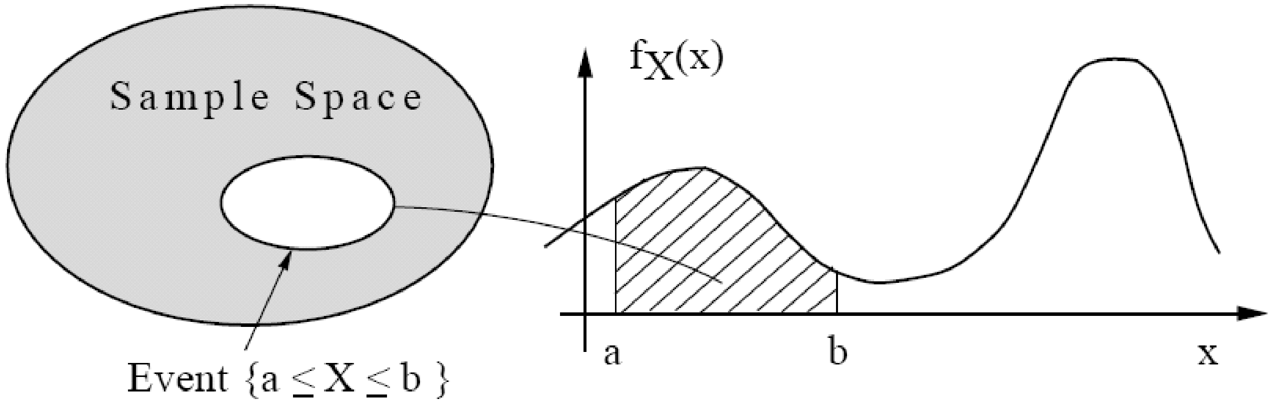
LECTURE 9

- Readings: Section 3.4-3.5

Lecture outline

- PDF: Review
- Multiple random variables
  - Conditioning
  - Independence
- Examples

Continuous r.v.'s and PDFs

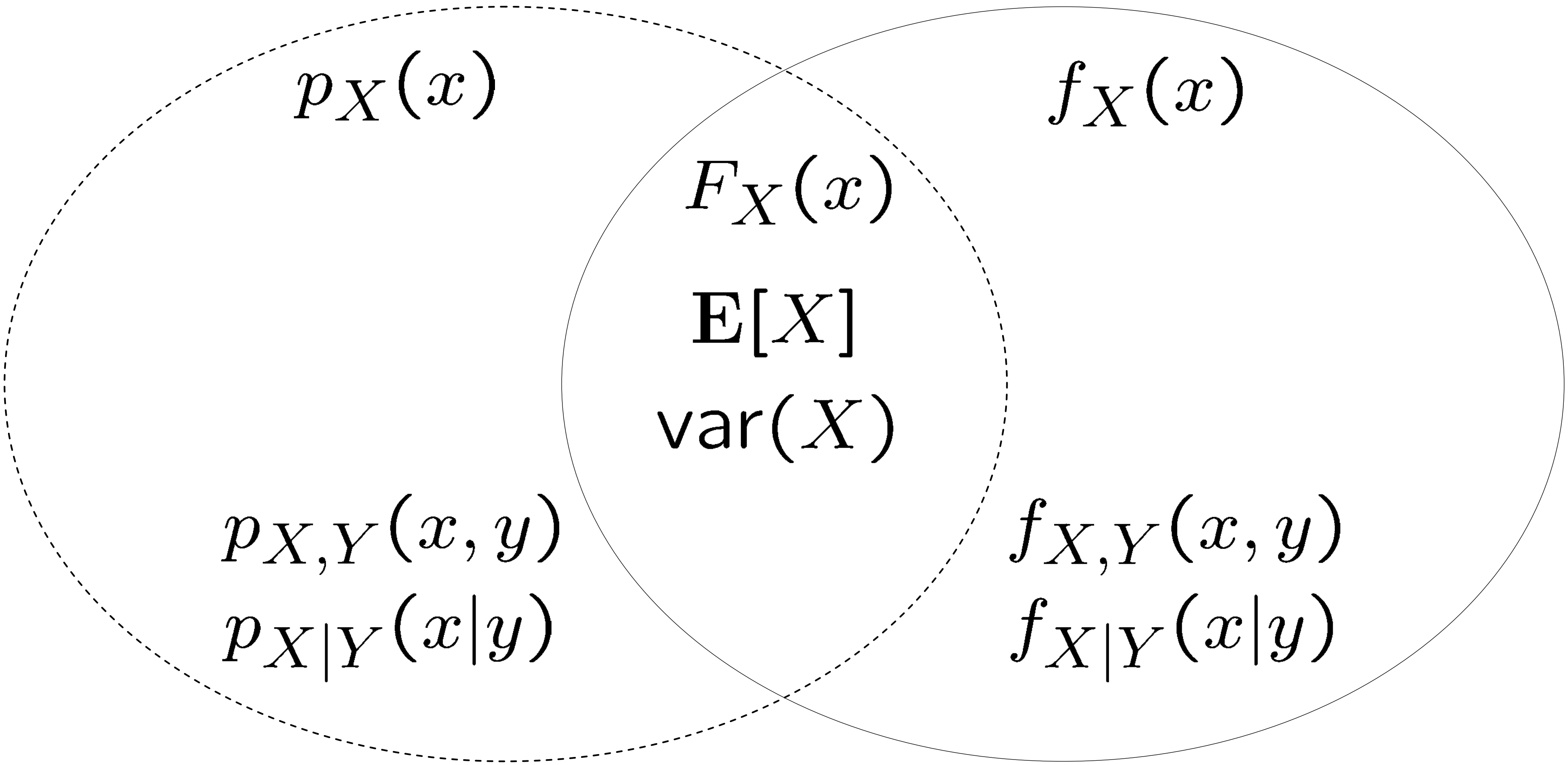


$$\mathbf{P}(a \leq X \leq b) = \int_a^b f_X(x)dx$$

- $\mathbf{P}(x \leq X \leq x + \delta) \approx f_X(x) \cdot \delta$
- $\mathbf{E}[g(X)] = \int_{-\infty}^{\infty} g(x) \cdot f_X(x)dx$

Summary of Concepts

Discrete **Continuous**



Joint PDF $f_{X,Y}(x,y)$ (1)

$$\mathbf{P}(A) = \iint_A f_{X,Y}(x,y)dxdy$$

- Interpretation:

$$\mathbf{P}(x \leq X \leq x+\delta, y \leq Y \leq y+\delta) \approx f_{X,Y}(x,y) \cdot \delta^2$$

- Expectation:

$$\mathbf{E}[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)f_{X,Y}(x,y)dxdy$$

Joint PDF $f_{X,Y}(x,y)$ (2)

$$\mathbf{P}(A) = \iint_A f_{X,Y}(x,y)dxdy$$

- From the joint to the marginal:

$$f_X(x) \cdot \delta \approx \mathbf{P}(x \leq X \leq x + \delta) =$$

$$\int_{-\infty}^{\infty} \int_x^{x+\delta} f_{X,Y}(t,y)dtdy \approx \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy \cdot \delta$$

- X and Y are called independent iff:

$$f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$$

Conditioning

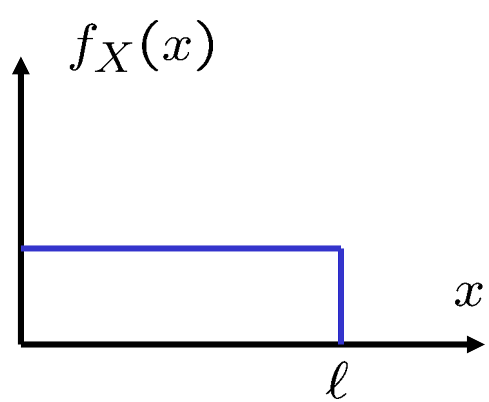
- Recall, again: $\mathbf{P}(x \le X \le x + \delta) \approx f_X(x) \cdot \delta$
- By analogy:

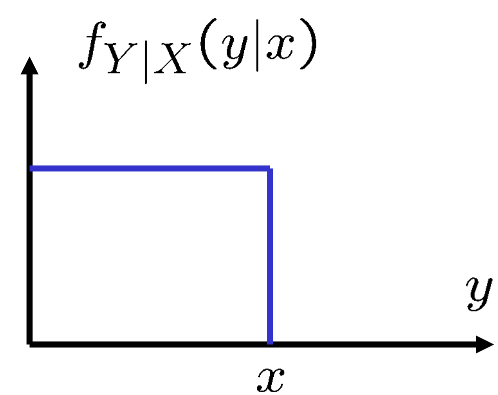
$$\mathbf{P}(x \le X \le x + \delta | Y \approx y) \approx f_{X|Y}(x|y) \cdot \delta$$

- Thus, the definition: $f_{X|Y}(x|y) = \dfrac{f_{X,Y}(x,y)}{f_Y(y)}$
- Conditioning is a "section" of the joint PDF, normalized.
- Independence gives: $f_{X|Y}(x|y) = f_X(x)$

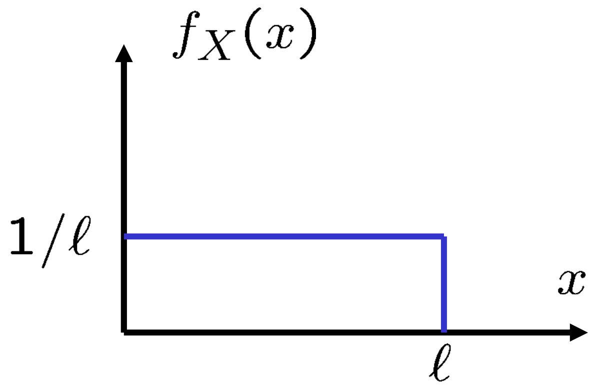
Example: Stick-Breaking (1)

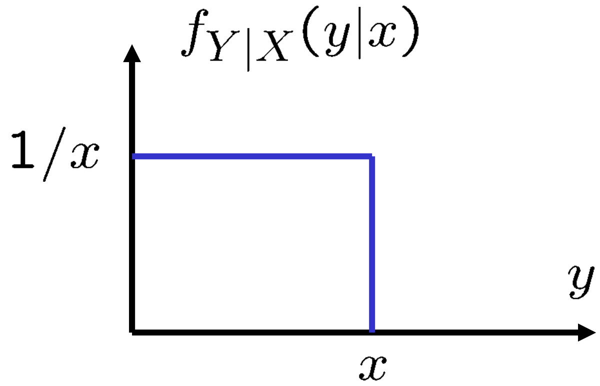
- Break a stick of length ℓ twice:
  - X: first break point, chosen uniformly between 0 and ℓ.
  - Y: second break point, chosen (given X=x) uniformly from 0 to x.





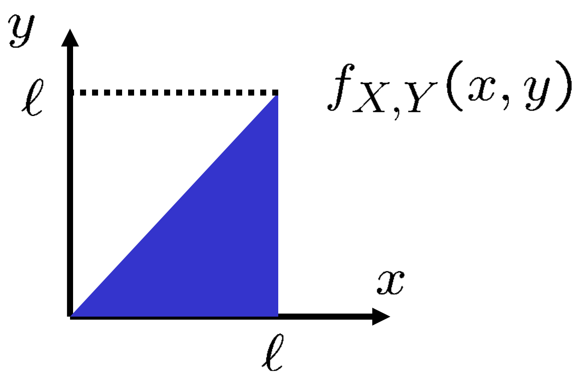
Stick-Breaking (2)





- Joint PDF: $f_{X,Y}(x,y) = f_X(x) \cdot f_{Y|X}(y|x)$

$$= \frac{1}{\ell x} \qquad 0 \le y < x \le \ell$$



Stick-Breaking (3)

- Conditional Expectation of Y, given X=x:

$$\mathbf{E}[Y \mid X = x] = \int y f_{Y|X}(y \mid X = x)\, dy = \frac{x}{2}$$

- Expectation of Y: $\mathbf{E}[Y] = \int_0^{\ell} y f_Y(y)\, dy$

$$\begin{aligned} f_Y(y) &= \int f_{X,Y}(x, y)\, dx \\ &= \int_y^{\ell} \frac{1}{\ell x}\, dx = \frac{1}{\ell} \log \frac{\ell}{y}, \qquad 0 \le y \le \ell \end{aligned}$$

$$\mathbf{E}[Y] = \int_0^{\ell} y \frac{1}{\ell} \log \frac{\ell}{y}\, dy = \frac{\ell}{4}$$