Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Spring 2006)

Problem Set 10
Topics: Poisson, Markov chains
Due: May 10th, 2006

1. (a) We are given that the previous ship to pass the pointer was traveling westward.
   i. The direction of the next ship is independent of those of any previous ships. Therefore, we are simply looking for the probability that a westbound arrival occurs before an eastbound arrival, or

$$\mathbf{P}(\text{next} = \text{westbound}) = \frac{\lambda_W}{\lambda_E + \lambda_W}$$

   ii. The pointer will change directions on the next arrival of an east-bound ship. By definition of the Poisson process, the remaining time until this arrival, denote it by X, is exponential with parameter λ_E, or

$$f_X(x) = \lambda_E e^{-\lambda_E x}\ , x \geq 0 \quad .$$

   (b) For this to happen, no westbound ship can enter the channel from the moment the eastbound ship enters until the moment it exits, which consumes an amount of time t after the eastward ship enters the channel. In addition, no westbound ships may already be in the channel prior to the eastward ship entering the channel, which requires that no westbound ships enter for an amount of time t *before* the eastbound ship enters. Together, we require no westbound ships to arrive during an interval of time $2t$, which occurs with probability

$$\frac{(\lambda_W 2t)^0 e^{-\lambda_W 2t}}{0!} = e^{-\lambda_W 2t} \quad .$$

   (c) Letting X be the first-order interarrival time for eastward ships, we can express the quantity $V = X_1 + X_2 + \ldots + X_7$, and thus the PDF for V is equivalent to the 7th order Erlang distribution

$$f_V(v) = \frac{\lambda_E^7 v^6 e^{-\lambda_E v}}{6!}\ , v \geq 0 \quad .$$

2. (a) The mean of a Poisson random variable with paramater λ is λ.

$$E[\text{number of passengers on a shuttle}] = \lambda$$

   (b) Let A be the number of shuttles arriving in one hour. The Poisson parameter now becomes $\mu \cdot 1$.

$$p_A(a) = \begin{cases} \frac{e^{-\mu}\mu^a}{a!} & a = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

   (c) To join Poisson process, the arrival rates are summed. The Poisson parameter now becomes $\lambda + \mu$.

$$E[\text{number of events per hour}] = \lambda + \mu$$

   (d) The Poisson process is memoryless, and the expected time until the first arrival for the shuttle bus is $\frac{1}{\mu}$.

$$E[\text{wait time} \mid 2\lambda \text{ people waiting}] = \frac{1}{\mu}$$

(e) Let N be the number of people on a shuttle. The number of people that are on the shuttle are the number of people that arrive before the first shuttle arrives. The PMF is a shifted geometric starting at 0:

$$p_N(n) = \left(\frac{\lambda}{\lambda+\mu}\right)^n \left(\frac{\mu}{\lambda+\mu}\right) \quad \text{for } n = 0, 1, 2, \ldots$$

3. (a) Let A_k be the event that the process enters S_2 for first time on trial k. The only way to enter state S_2 for the first time on the kth trial is to enter state S_3 on the first trial, remain in S_3 for the next $k-2$ trials, and finally enter S_2 on the last trial. Thus,

$$\mathbf{P}(A_k) = p_{03} \cdot p_{33}^{k-2} \cdot p_{32} = \left(\frac{1}{3}\right)\left(\frac{1}{4}\right)^{k-2}\left(\frac{1}{4}\right) = \frac{1}{3}\left(\frac{1}{4}\right)^{k-1} \qquad \text{for} \quad k = 2, 3, \ldots$$

(b) Let A be the event that the process never enters S_4.
There are three possible ways for A to occur. The first two are if the first transition is either from S_0 to S_1 or S_0 to S_5. This occurs with probability $\frac{2}{3}$. The other is if The first transition is from S_0 to S_3, and that the next change of state *after* that is to the state S_2. We know that the probability of going from S_0 to S_3 is $\frac{1}{3}$. Given this has occurred, and given a change of state occurs from state S_3, we know that the probability that the state transitioned to is the state S_2 is simply $\frac{\frac{1}{4}}{\frac{1}{4}+\frac{1}{2}} = \frac{1}{3}$. Thus, the probability of transitioning from S_0 to S_3 and then eventually transitioning to S_2 is $\frac{1}{9}$. Thus, the probability of never entering S_4 is $\frac{2}{3} + \frac{1}{9} = \frac{7}{9}$.

(c) $\mathbf{P}(\{\text{process enters } S_2 \text{ and then leaves } S_2 \text{ on next trial}\})$

$$\begin{aligned}
&= \mathbf{P}(\{\text{process enters } S_2\})\mathbf{P}(\{\text{leaves } S_2 \text{ on next trial }\}|\{\text{ in } S_2\}) \\
&= \left[\sum_{k=2}^{\infty} \mathbf{P}(A_k)\right] \cdot \frac{1}{2} \\
&= \left[\sum_{k=2}^{\infty} \frac{1}{3}(\frac{1}{4})^{k-1}\right] \cdot \frac{1}{2} \\
&= \frac{1}{6} \cdot \frac{\frac{1}{4}}{1-\frac{1}{4}} \\
&= \frac{1}{18}.
\end{aligned}$$

(d) This event can only happen if the sequence of state transitions is as follows:

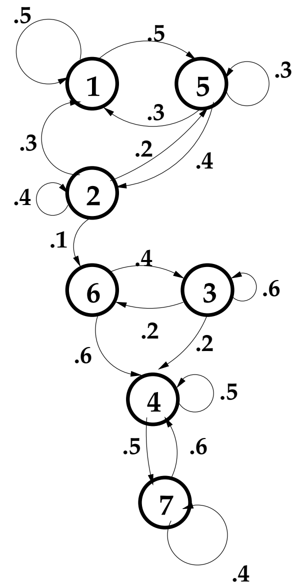
$$S_0 \longrightarrow S_3 \longrightarrow S_2 \longrightarrow S_1.$$

Thus, $\mathbf{P}(\{\text{process enters } S_1 \text{ for first time on third trial}\}) = p_{03} \cdot p_{32} \cdot p_{21} = \frac{1}{3} \cdot \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{24}$.

(e) $\mathbf{P}(\{\text{process in } S_3 \text{ immediately after the } N\text{th trial}\})$

$$\begin{aligned}
&= \mathbf{P}(\{\text{moves to } S_3 \text{ in first trial and stays in } S_3 \text{ for next } N-1 \text{ trials}\}) \\
&= \frac{1}{3}\left(\frac{1}{4}\right)^{n-1} \qquad \text{for } n = 1, 2, 3, \ldots
\end{aligned}$$

4. Let $i, i = 1, \ldots, 7$ be the states of the Markov chain. From the graphical representation of the transition matrix it is easy to see the following:



   (a) $\{4, 7\}$ are recurrent and the rest are transient.

   (b) There is only one class formed by the recurrent states.

   (c) Since state 1 is not reachable from state 4, $\lim_{n\to\infty} p_{41}(n) = 0$. Since 6 is a transient state $\lim_{n\to\infty} p_{66}(n) = 0$.

5. (a) Given L_{n-1}, the history of the process (i.e., $L_{n-2}, L_{n-3}, \ldots$) is irrelevant for determining the probability distribution of L_n, the number of remaining unlocked doors at time n. Therefore, L_n is Markov. More precisely,

$$\mathbf{P}(L_n = j | L_{n-1} = i, L_{2-2} = k, \ldots, L_1 = q) = \mathbf{P}(L_n = j | L_{n-1} = i) = p_{ij}.$$

   Clearly, at one step the number of unlocked doors can only decrease by one or stay constant. So, for $1 \le i \le d$, if $j = i-1$, then $p_{ij} = \mathbf{P}$(selecting an unlocked door on day $n+1 | L_n = i) = \frac{i}{d}$. For $0 \le i \le d$, if $j = i$, then $p_{ij} = \mathbf{P}$(selecting an locked door on day $n+1 | L_n = i) = \frac{d-i}{d}$. Otherwise, $p_{ij} = 0$. To summarize, for $0 \le i, j \le d$, we have the following:

$$p_{ij} = \begin{cases} \frac{d-i}{d} & j = i \\ \frac{i}{d} & j = i-1 \\ 0 & \text{otherwise} \end{cases}$$

   (b) The state with 0 unlocked doors is the only recurrent state. All other states are then transient, because from each, there is a positive probability of going to state 0, from which it is not possible to return.

   (c) Note that once all the doors are locked, none will ever be unlocked again. So the state 0 is an absorbing state: there is a positive probability that the system will enter it, and once it does, it will remain there forever. Then, clearly, $\lim_{n\to\infty} r_{i0}(n) = 1$ and $\lim_{n\to\infty} r_{ij}(n) = 0$ for all $j \ne 0$ and all i, .

(d) Now, if I choose a locked door, the number of unlocked doors will increase by one the next day. Similarly, the number of unlocked doors will decrease by 1 if and only if I choose an unlocked door. Hence,

$$p_{ij} = \begin{cases} \frac{d-i}{d} & j = i+1 \\ \frac{i}{d} & j = i-1 \\ 0 & \text{otherwise} \end{cases}$$

Clearly, from each state one can go to any other state and return with positive probability, hence all the states in this Markov chain communicate and thus form one recurrent class. There are no transient states or absorbing states. Note however, that from an even-numbered state (states 0, 2, 4, etc) one can only go to an odd-numbered state in one step, and similarly all one-step transitions from odd-numbered states lead to even-numbered states. Since the states can be grouped into two groups such that all transitions from one lead to the other and vice versa, the chain is periodic with period 2. This will lead to $r_{ij}(n)$ oscillating and not converging as $n \to \infty$. For example, $r_{11}(n) = 0$ for all odd n, but positive for even n.

(e) In this case L_n is not a Markov process. To see this, note that $\mathbf{P}(L_n = i+1 | L_{n-1} = i, L_{n-2} = i-1) = 0$ since according to my strategy I do not unlock doors two days in a row. But clearly, $\mathbf{P}(L_n = i+1 | L_{n-1} = i) > 0$ for $i < d$ since it is possible to go from a state of i unlocked doors to a state of $i+1$ unlocked doors in general. Thus $\mathbf{P}(L_n = i+1 | L_{n-1} = i, L_{n-2} = i-1) \neq \mathbf{P}(L_n = i+1 | L_{n-1} = i)$, which shows that L_n does not have the Markov property.

G1†. a) First let the p_{ij}'s be the transition probabilities of the Markov chain.

Then

$$\begin{aligned}
m_{k+1}(1) &= E[R_{k+1} | X_0 = 1] \\
&= E[g(X_0) + g(X_1) + ... + g(X_{k+1}) | X_0 = 1] \\
&= \sum_{i=1}^{n} p_{1i} E[g(X_0) + g(X_1) + ... + g(X_{k+1}) | X_0 = 1, X_1 = i] \\
&= \sum_{i=1}^{n} p_{1i} E[g(1) + g(X_1) + ... + g(X_{k+1}) | X_1 = i] \\
&= g(1) + \sum_{i=1}^{n} p_{1i} E[g(X_1) + ... + g(X_{k+1}) | X_1 = i] \\
&= g(1) + \sum_{i=1}^{n} p_{1i} m_k(i)
\end{aligned}$$

and thus in general $m_{k+1}(c) = g(c) + \sum_{i=1}^{n} p_{ci} m_k(i)$ when $c \in \{1, ..., n\}$.

Note that the third equality simply uses the total expectation theorem.

b)

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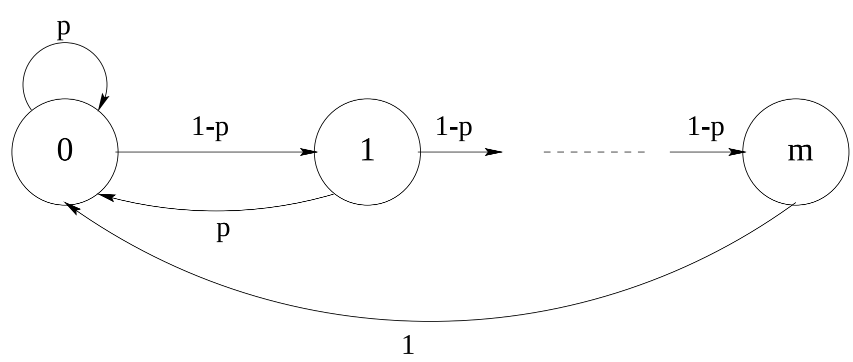
\begin{aligned}
v_{k+1}(1) &= Var[R_{k+1}|X_0 = 1] \\
&= Var[g(X_0) + g(X_1) + ... + g(X_{k+1})|X_0 = 1] \\
&= Var[E[g(X_0) + g(X_1) + ... + g(X_{k+1})|X_0 = 1, X_1]] + \\
&\quad E[Var[g(X_0) + g(X_1) + ... + g(X_{k+1})|X_0 = 1, X_1]] \\
&= Var[g(1) + E[g(X_1) + ... + g(X_{k+1})|X_0 = 1, X_1]] + \\
&\quad E[Var[g(1) + g(X_1) + ... + g(X_{k+1})|X_0 = 1, X_1]] \\
&= Var[E[g(X_1) + ... + g(X_{k+1})|X_0 = 1, X_1]] + E[Var[g(X_1) + ... + g(X_{k+1})|X_0 = 1, X_1]] \\
&= Var[E[g(X_1) + ... + g(X_{k+1})|X_1]] + E[Var[g(X_1) + ... + g(X_{k+1})|X_1]] \\
&= Var[m_k(X_1)] + E[v_k(X_1)] \\
&= E[(m_k(X_1))^2] - E[m_k(X_1)]^2 + \sum_{i=1}^{n} p_{1i} v_k(i) \\
&= \sum_{i=1}^{n} p_{1i} m_k^2(i) - (\sum_{i=1}^{n} p_{1i} m_k(i))^2 + \sum_{i=1}^{n} p_{1i} v_k(i)
\end{aligned}
$$

so in general $v_{k+1}(c) = \sum_{i=1}^{n} p_{ci} m_k^2(i) - (\sum_{i=1}^{n} p_{ci} m_k(i))^2 + \sum_{i=1}^{n} p_{ci} v_k(i)$ when $c \in \{1, ..., n\}$.

G2†. We introduce the states $0, 1, \ldots, m$ and identify them as the number of days the gate survives a crash. The state transition diagram is shown in the figure below.



The balance equations take the form,

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\begin{aligned}
\pi_0 &= \pi_0 p + \pi_1 p + \cdots + \pi_{m-1} p + \pi_m \\
\pi_1 &= \pi_0 (1-p) \\
\pi_2 &= \pi_1 (1-p) = \pi_0 (1-p)^2 \\
&\vdots \\
\pi_m &= \pi_0 (1-p)^m
\end{aligned}
$$

These equations together with the normalization equation have a unique solution which gives us the steady-state probabilities of all states. The steady-state expected frequency of gate replacements is the expected frequency of visits to state 0, which by frequency interpretation is given by π_0. Solving the above equations with the normalization equation we get,

$$\begin{aligned} E[\text{frequency of gate replacements}] &= \pi_0 \\ &= \frac{p}{1-(1-p)^{m+1}} \end{aligned}$$