Problem Set 1
Due: February 15, 2006

1. Express each of the following events in terms of the events A, B and C as well as the operations of complementation, union and intersection:

   (a) at least one of the events A, B, C occurs;
   (b) at most one of the events A, B, C occurs;
   (c) none of the events A, B, C occurs;
   (d) all three events A, B, C occur;
   (e) exactly one of the events A, B, C occurs;
   (f) events A and B occur, but not C;
   (g) either event A occurs or, if not, then B also does not occur.

   In each case draw the corresponding Venn diagrams.

2. Let A and B be two events. Use the axioms of probability to prove the following:

   (a) $P(A \cap B) \geq P(A) + P(B) - 1$
   (b) Show that the probability that one and only one of the events A or B occurs is $P(A) + P(B) - 2 \cdot P(A \cap B)$.

   Note: You may want to argue in terms of Venn diagrams, but you should also provide a complete proof, that is a step-by-step derivation, where each step appeals to an axiom or a logical rule.

3. Find $P\left(A \cup (B^c \cup C^c)^c\right)$ in each of the following cases:

   (a) A, B, C are mutually exclusive events and $P(A) = 3/7$.
   (b) $P(A) = 1/2$, $P(B \cap C) = 1/3$, $P(A \cap C) = 0$.
   (c) $P\left(A^c \cap (B^c \cup C^c)\right) = 0.65$.

4. Anne and Bob each have a deck of playing cards. Each flips over a randomly selected card. Assume that all pairs of cards are equally likely to be drawn. Determine the following probabilities:

   (a) the probability that at least one card is an ace,
   (b) the probability that the two cards are of the same suit,
   (c) the probability that neither card is an ace,
   (d) the probability that neither card is a diamond or club.

5. Alice and Bob each choose at random a number between zero and two. We assume a uniform probability law under which the probability of an event is proportional to its area. Consider the following events:

   A : The magnitude of the difference of the two numbers is greater than 1/3.

B : At least one of the numbers is greater than 1/3.

C : The two numbers are equal.

D : Alice's number is greater than 1/3.

Find the probabilities $P(B)$, $P(C)$, $P(A \cap D)$.

6. Bob has a peculiar pair of four-sided dice. When he rolls the dice, the probability of any particular outcome is proportional to the product of the outcome of each die. All outcomes that result in a particular product are equally likely.

   (a) What is the probability of the product being even?

   (b) What is the probability of Bob rolling a 2 and a 3?

G1†. Let $A, B, C, A_1, \ldots, A_n$ be some events. Show the following identities. A mathematical derivation is required, but you can use Venn diagrams to guide your thinking.

   (a) $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$,

   (b) $P(\cup_{k=1}^{n} A_k) = P(A_1) + P(A_1^c \cap A_2) + P(A_1^c \cap A_2^c \cap A_3) + \cdots + P(A_1^c \cap \cdots \cap A_{n-1}^c \cap A_n)$.

†Required for 6.431; optional for 6.041