LECTURE 9

- **Readings:** Sections 3.4-3.5

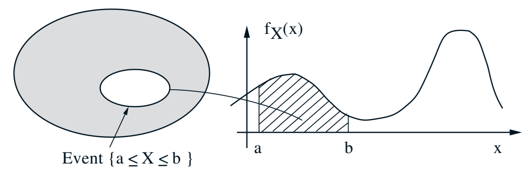
Outline

- PDF review
- Multiple random variables
  - conditioning
  - independence
- Examples

Summary of concepts

$p_X(x)$		$f_X(x)$
	$F_X(x)$	
$\sum_x x p_X(x)$	$\mathbf{E}[X]$	$\int x f_X(x)\,dx$
	$\text{var}(X)$	
$p_{X,Y}(x,y)$		$f_{X,Y}(x,y)$
$p_{X\|A}(x)$		$f_{X\|A}(x)$
$p_{X\|Y}(x \mid y)$		$f_{X\|Y}(x \mid y)$

Continuous r.v.'s and pdf's



$$\mathbf{P}(a \le X \le b) = \int_a^b f_X(x)\,dx$$

- $\mathbf{P}(x \le X \le x+\delta) \approx f_X(x) \cdot \delta$
- $\mathbf{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx$

Joint PDF $f_{X,Y}(x,y)$

$$\mathbf{P}((X,Y) \in S) = \int\int_S f_{X,Y}(x,y)\,dx\,dy$$

- Interpretation:

$$\mathbf{P}(x \le X \le x+\delta,\ y \le Y \le y+\delta) \approx f_{X,Y}(x,y) \cdot \delta^2$$

- Expectations:

$$\mathbf{E}[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)\,dx\,dy$$

- From the joint to the marginal:

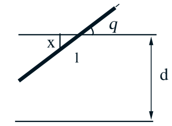
$$f_X(x) \cdot \delta \approx \mathbf{P}(x \le X \le x+\delta) =$$

- X and Y are called independent if

$$f_{X,Y}(x,y) = f_X(x) f_Y(y), \qquad \text{for all } x, y$$

Buffon's needle

- Parallel lines at distance d
  Needle of length ℓ (assume $\ell < d$)
- Find $\mathbf{P}$(needle intersects one of the lines)



- $X \in [0, d/2]$: distance of needle midpoint to nearest line
- **Model:** X, Θ uniform, independent

$$f_{X,\Theta}(x,\theta) = \qquad 0 \le x \le d/2,\ 0 \le \theta \le \pi/2$$

- Intersect if $X \le \frac{\ell}{2} \sin \Theta$

$$\begin{aligned}
\mathbf{P}\left(X \le \frac{\ell}{2}\sin\Theta\right) &= \int\int_{x \le \frac{\ell}{2}\sin\theta} f_X(x) f_\Theta(\theta)\,dx\,d\theta \\
&= \frac{4}{\pi d}\int_0^{\pi/2}\int_0^{(\ell/2)\sin\theta} dx\,d\theta \\
&= \frac{4}{\pi d}\int_0^{\pi/2} \frac{\ell}{2}\sin\theta\,d\theta = \frac{2\ell}{\pi d}
\end{aligned}$$

Conditioning

- Recall

$$\mathbf{P}(x \le X \le x+\delta) \approx f_X(x)\cdot\delta$$

- By analogy, would like:

$$\mathbf{P}(x \le X \le x+\delta \mid Y \approx y) \approx f_{X|Y}(x \mid y)\cdot\delta$$

- This leads us to the **definition**:

$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} \qquad \text{if } f_Y(y) > 0$$

- For given y, conditional PDF is a (normalized) "section" of the joint PDF

- If independent, $f_{X,Y} = f_X f_Y$, we obtain

$$f_{X|Y}(x|y) = f_X(x)$$

Joint, Marginal and Conditional Densities

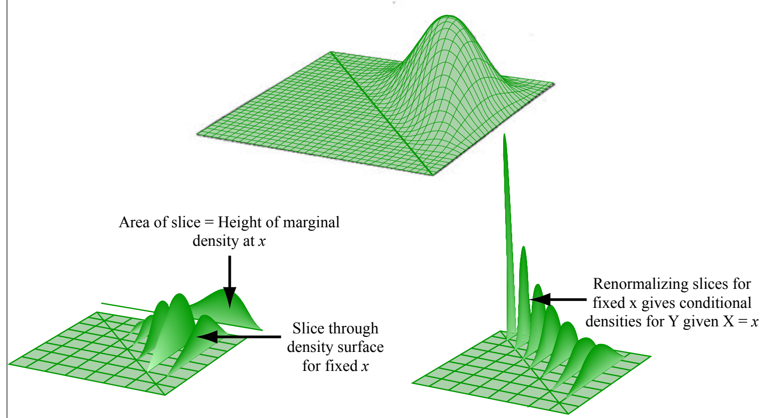
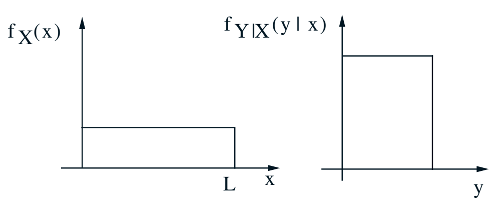


Image by MIT OpenCourseWare, adapted from *Probability*, by J. Pittman, 1999.

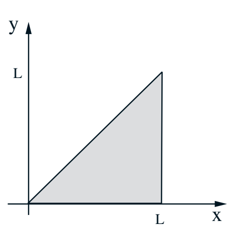
Stick-breaking example

- Break a stick of length ℓ twice:
  break at X: uniform in $[0,1]$;
  break again at Y, uniform in $[0,X]$



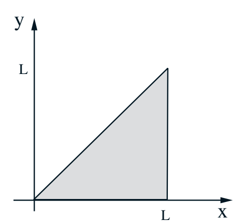
$$f_{X,Y}(x,y) = f_X(x) f_{Y|X}(y \mid x) =$$

on the set:



$$\mathbf{E}[Y \mid X = x] = \int y f_{Y|X}(y \mid X = x)\,dy =$$

$$f_{X,Y}(x,y) = \frac{1}{\ell x}, \qquad 0 \le y \le x \le \ell$$



$$\begin{aligned} f_Y(y) &= \int f_{X,Y}(x,y)\,dx \\ &= \int_y^\ell \frac{1}{\ell x}\,dx \\ &= \frac{1}{\ell}\log\frac{\ell}{y}, \qquad 0 \le y \le \ell \end{aligned}$$

$$\mathbf{E}[Y] = \int_0^\ell y f_Y(y)\,dy = \int_0^\ell y \frac{1}{\ell}\log\frac{\ell}{y}\,dy = \frac{\ell}{4}$$