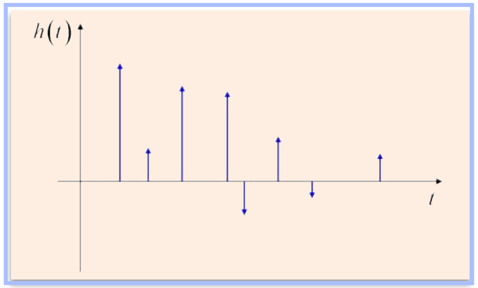
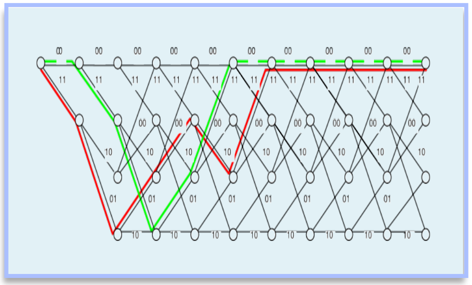
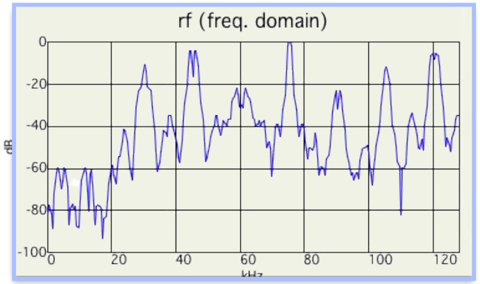
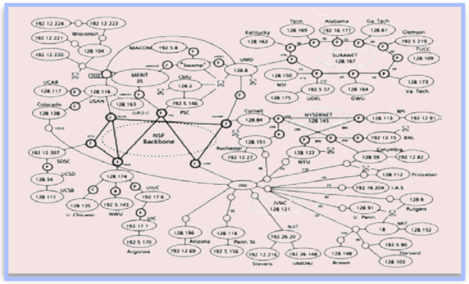








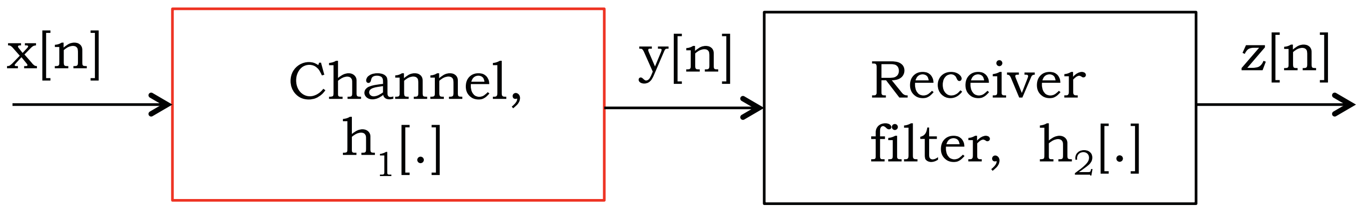
INTRODUCTION TO EECS II

DIGITAL COMMUNICATION SYSTEMS

6.02 Fall 2012
Lecture #14

- Spectral content via the DTFT

Demo: "Deconvolving" Output of Channel with Echo



Suppose channel is LTI with

$$h_1[n]=\delta[n]+0.8\delta[n-1]$$

$$H_1(\Omega) = ?? \quad = \boxed{\sum_m h_1[m]e^{-j\Omega m}}$$

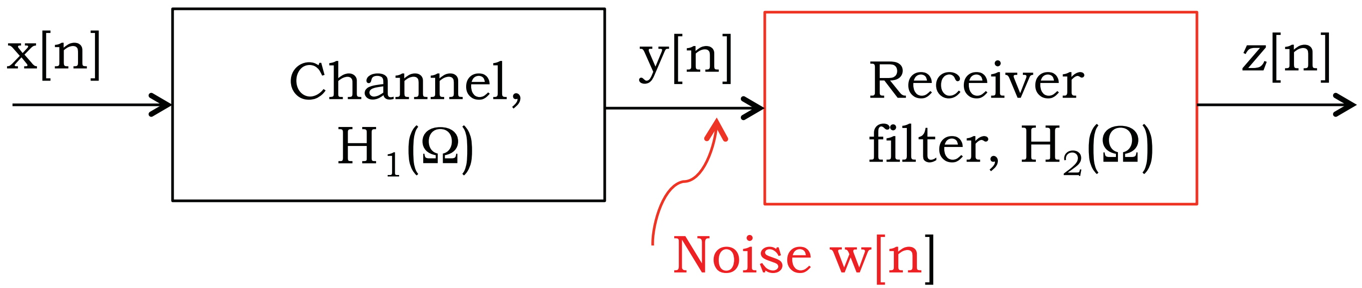
$$= 1 + 0.8e^{-j\Omega} = 1 + 0.8\cos(\Omega) - j0.8\sin(\Omega)$$

So:

$$|H_1(\Omega)| = [1.64 + 1.6\cos(\Omega)]^{1/2}$$ *EVEN function of* Ω;

$$<H_1(\Omega) = \arctan\left[-(0.8\sin(\Omega)/[1 + 0.8\cos(\Omega)]\right]$$ *ODD*.

A Frequency-Domain view of Deconvolution



Given $H_1(\Omega)$, what should $H_2(\Omega)$ be, to get z[n]=x[n]?

$\Rightarrow$ $H_2(\Omega) = 1/H_1(\Omega)$ "Inverse filter"

$$= (1/|H_1(\Omega)|).\ \exp\{-j<H_1(\Omega)\}$$

Inverse filter at receiver does very badly in the presence of noise that adds to y[n]:

filter has high gain for noise precisely at frequencies where channel gain $|H_1(\Omega)|$ is low (and channel output is weak)!

DT Fourier Transform (DTFT) for Spectral Representation of General x[n]

If we can write

$$h[n] = \frac{1}{2\pi} \int_{<2\pi>} H(\Omega) e^{j\Omega n} d\Omega \qquad \text{where} \qquad H(\Omega) = \sum_m h[m] e^{-j\Omega m}$$

Any contiguous interval of length 2π

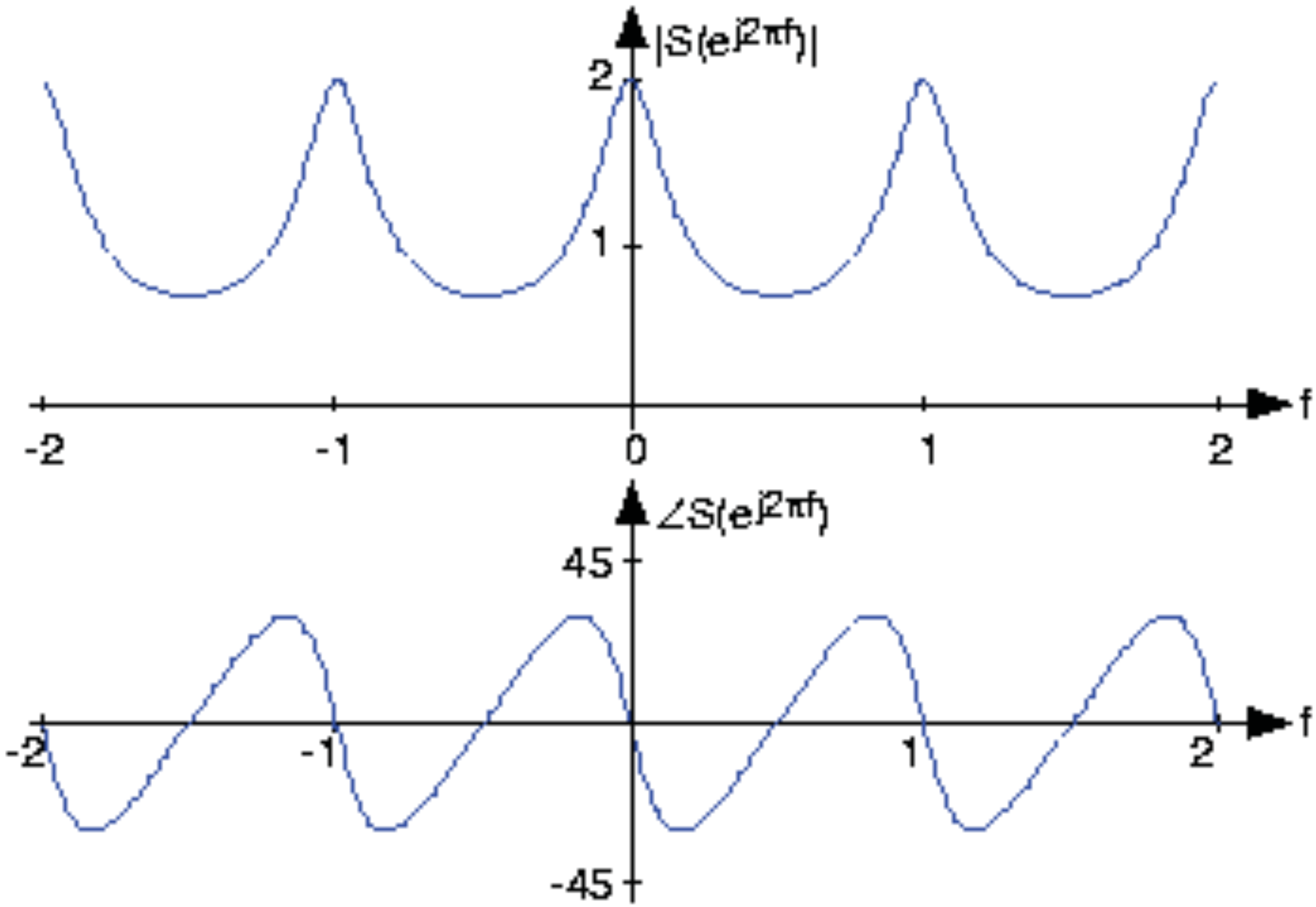
then we can write

$$x[n] = \frac{1}{2\pi} \int_{<2\pi>} X(\Omega) e^{j\Omega n} d\Omega \qquad \text{where} \qquad X(\Omega) = \sum_m x[m] e^{-j\Omega m}$$

This Fourier representation expresses x[n] as a weighted combination of $e^{j\Omega n}$ for all Ω in $[-\pi, \pi]$.

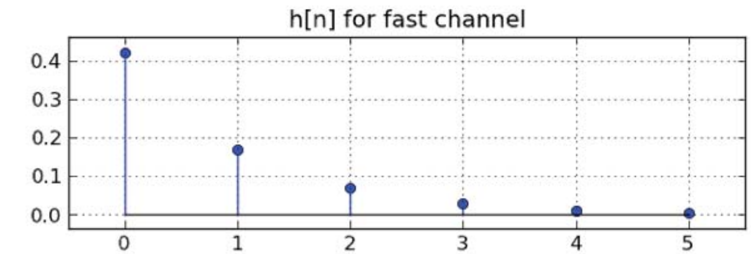
$X(\Omega_o)d\Omega$ is the **spectral content** of x[n] in the frequency interval $[\Omega_o, \Omega_o + d\Omega]$

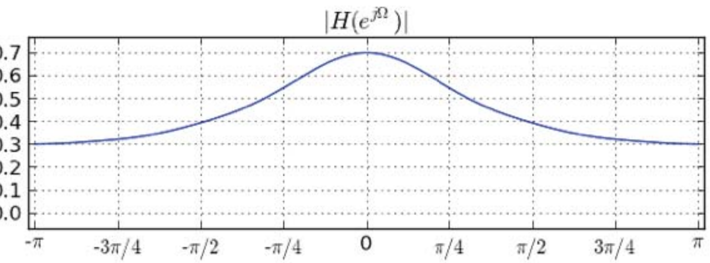
The spectrum of the exponential signal $(0.5)^n u[n]$ is shown over the frequency range $\Omega = 2\pi f$ in $[-4\pi, 4\pi]$, The angle has units of degrees.

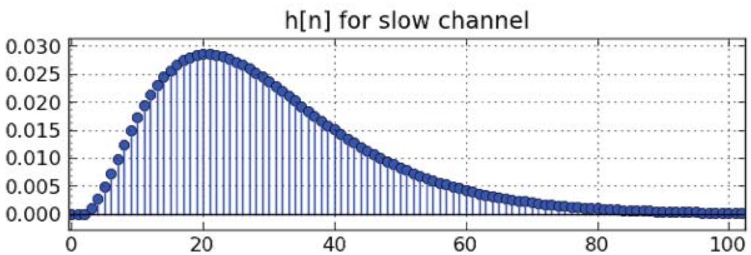


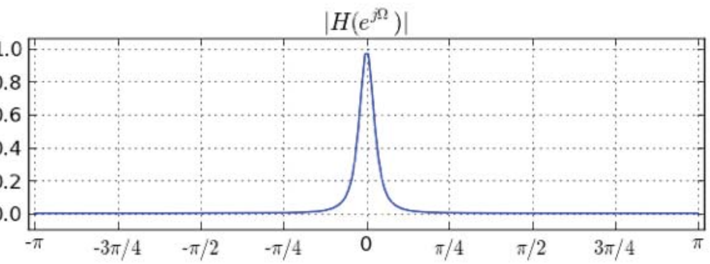
http://cnx.org/content/m0524/latest/

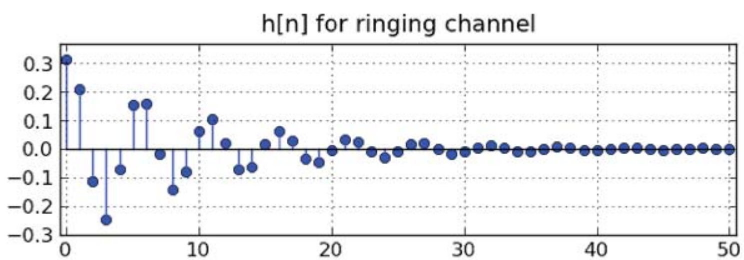
x[n] and X(Ω)

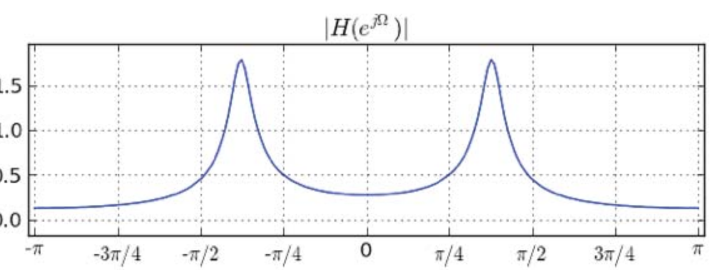

h[n] for fast channel
0.4
0.3
0.2
0.1
0.0
0
1
2
3
4
5


$|H(e^{j\Omega})|$
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
-π
-3π/4
-π/2
-π/4
0
π/4
π/2
3π/4
π


h[n] for slow channel
0.030
0.025
0.020
0.015
0.010
0.005
0.000
0
20
40
60
80
100


$|H(e^{j\Omega})|$
1.0
0.8
0.6
0.4
0.2
0.0
-π
-3π/4
-π/2
-π/4
0
π/4
π/2
3π/4
π


h[n] for ringing channel
0.3
0.2
0.1
0.0
−0.1
−0.2
−0.3
0
10
20
30
40
50


$|H(e^{j\Omega})|$
1.5
1.0
0.5
0.0
-π
-3π/4
-π/2
-π/4
0
π/4
π/2
3π/4
π

Input/Output Behavior of LTI System in Frequency Domain

$$x[n] = \frac{1}{2\pi} \int_{<2\pi>} X(\Omega) e^{j\Omega n} d\Omega$$

$$\xrightarrow{} \boxed{H(\Omega)} \xrightarrow{}$$

$$y[n] = \frac{1}{2\pi} \int_{<2\pi>} H(\Omega) X(\Omega) e^{j\Omega n} d\Omega$$

$$y[n] = \frac{1}{2\pi} \int_{<2\pi>} Y(\Omega) e^{j\Omega n} d\Omega$$

$$Y(\Omega) = H(\Omega) X(\Omega)$$

Compare with y[n]=(h*x)[n]

Again, convolution in time has mapped to multiplication in frequency

Magnitude and Angle

$$Y(\Omega) = H(\Omega)X(\Omega)$$

$$\Updownarrow$$

$$|Y(\Omega)| = |H(\Omega)| . |X(\Omega)|$$

and

$$< Y(\Omega) = < H(\Omega) + < X(\Omega)$$

Core of the Story

1. A huge class of DT and CT signals
can be written --- using **Fourier** transforms --- as a **weighted sums of sinusoids** (ranging from very slow to very fast) or (equivalently, but more compactly) complex exponentials.
The sums can be discrete $\sum$ or continuous $\int$ (or both).

2. **LTI** systems act very simply on sums of sinusoids:
superposition of responses to each sinusoid, with the **frequency response** determining the frequency-dependent scaling of magnitude, shifting in phase.

Loudspeaker Bandpass Frequency Response

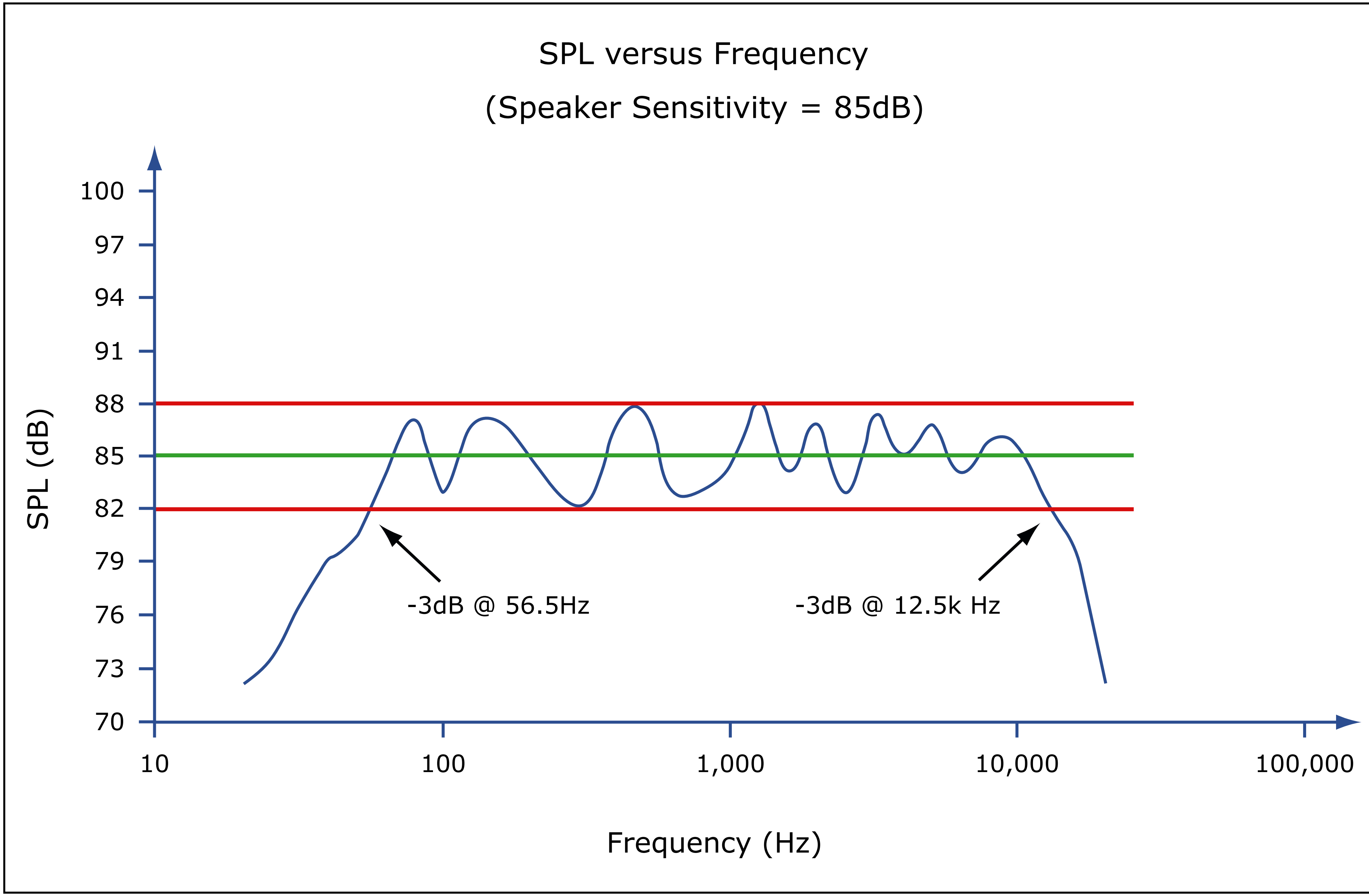
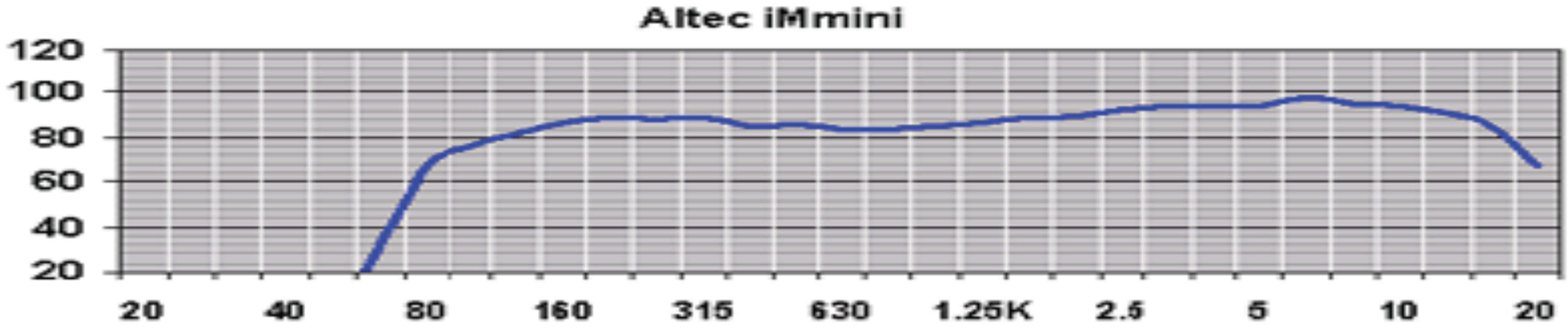
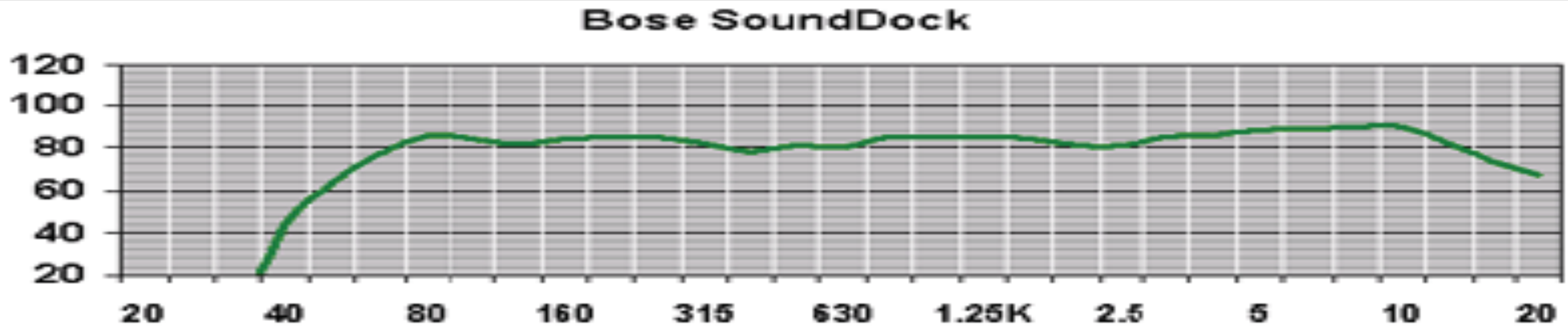
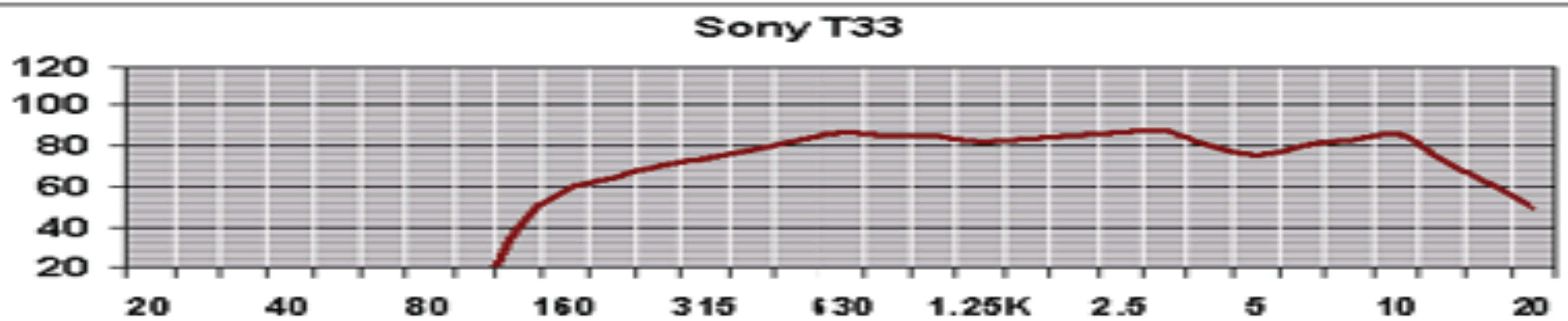


Image by MIT OpenCourseWare.


Altec iMmini
120
100
80
60
40
20
20
40
80
160
315
630
1.25K
2.5
5
10
20


Bose SoundDock
120
100
80
60
40
20
20
40
80
160
315
630
1.25K
2.5
5
10
20


Sony T33
120
100
80
60
40
20
20
40
80
160
315
630
1.25K
2.5
5
10
20

Spectral Content of Various Sounds

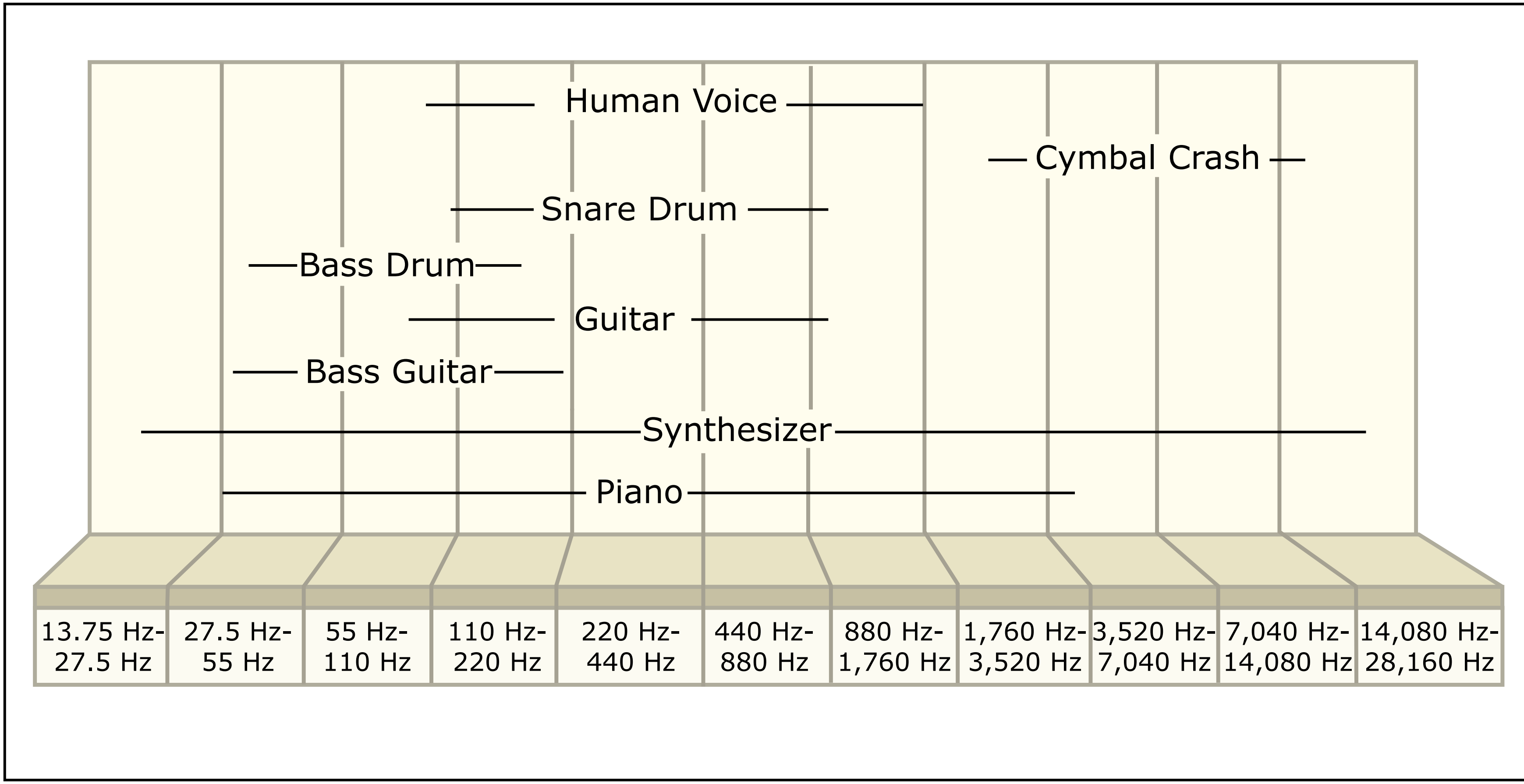


Image by MIT OpenCourseWare.

Connection between CT and DT

The continuous-time (CT) signal

$$x(t) = \cos(\omega t) = \cos(2\pi f t)$$

sampled every T seconds, i.e., at a sampling frequency of $f_s = 1/T$, gives rise to the discrete-time (DT) signal

$$x[n] = x(nT) = \cos(\omega n T) = \cos(\Omega n)$$

So $$\Omega = \omega T$$

and $\Omega = \pi$ corresponds to $\omega = \pi/T$ or $f = 1/(2T) = f_s/2$

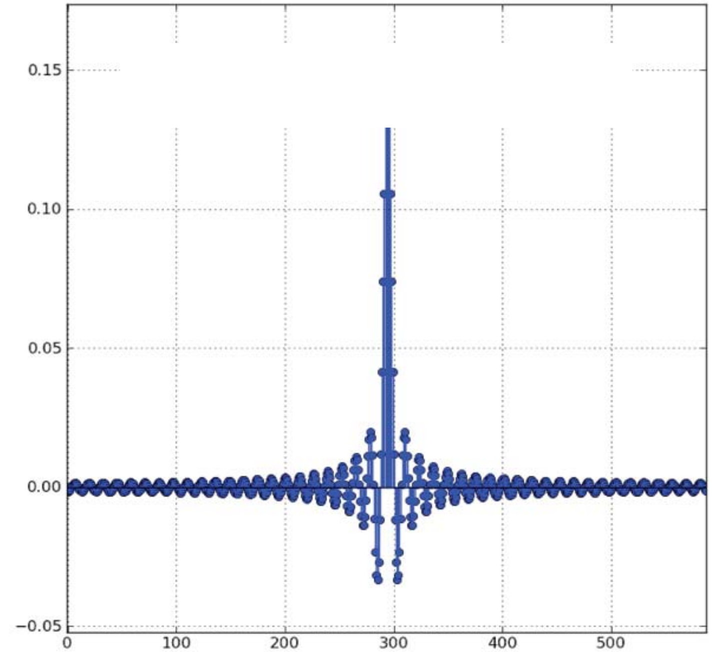
Signal x[n] that has its frequency content uniformly distributed in [-Ω_c , Ω_c]

$$x[n] = \frac{1}{2\pi} \int_{<2\pi>} X(\Omega) e^{j\Omega n} d\Omega$$

$$= \frac{1}{2\pi} \int_{-\Omega_C}^{\Omega_C} 1 \cdot e^{j\Omega n} d\Omega$$

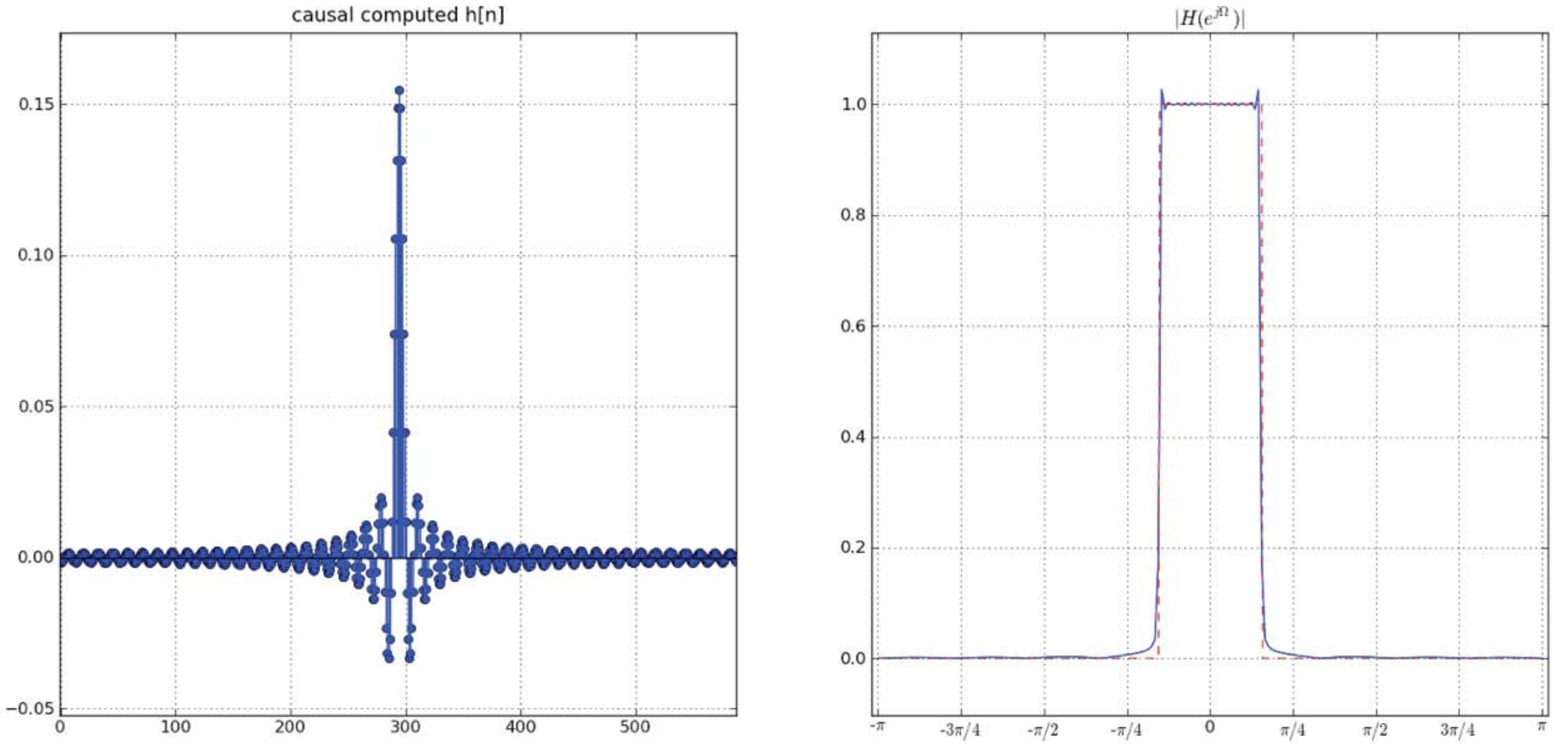
$$= \boxed{\frac{\sin(\Omega_C n)}{\pi n}}, \quad n \neq 0$$

$$= \Omega_C / \pi \qquad , \quad n = 0$$



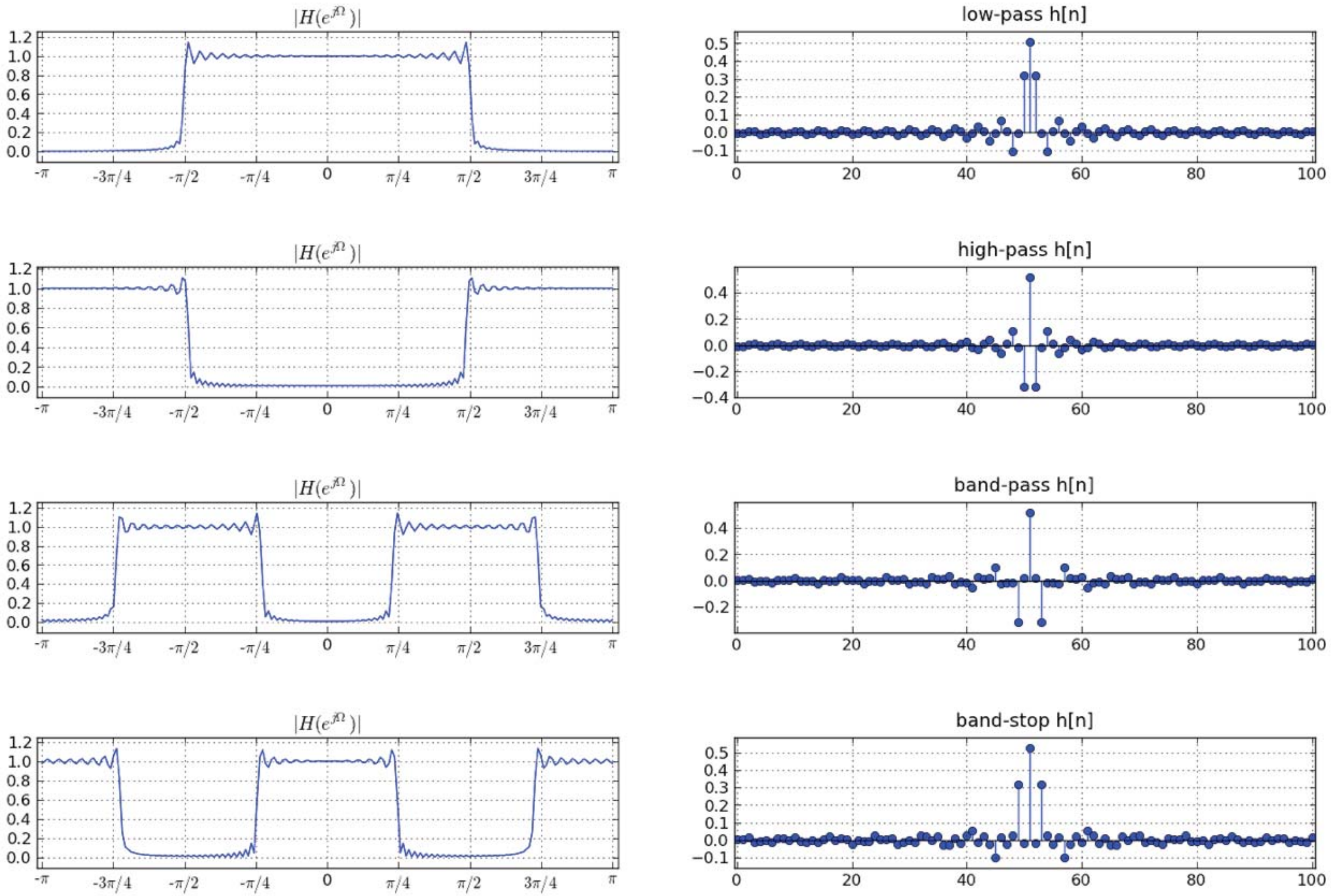
DT "sinc" function
(extends to ±∞ in time, falls off only as 1/n)

x[n] and X(Ω)



causal computed h[n]
0.15
0.10
0.05
0.00
−0.05
0
100
200
300
400
500
$|H(e^{j\Omega})|$
1.0
0.8
0.6
0.4
0.2
0.0
$-\pi$
$-3\pi/4$
$-\pi/2$
$-\pi/4$
0
$\pi/4$
$\pi/2$
$3\pi/4$
π

X(Ω) and x[n]



|H(e^{jΩ})|
low-pass h[n]
high-pass h[n]
band-pass h[n]
band-stop h[n]
-π -3π/4 -π/2 -π/4 0 π/4 π/2 3π/4 π
0 20 40 60 80 100

Fast Fourier Transform (FFT) to compute samples of the DTFT for signals of finite duration

$$X(\Omega_k) = \sum_{m=0}^{P-1} x[m] e^{-j\Omega_k m}, \qquad x[n] = \frac{1}{P} \sum_{k=-P/2}^{(P/2)-1} X(\Omega_k) e^{j\Omega_k n}$$

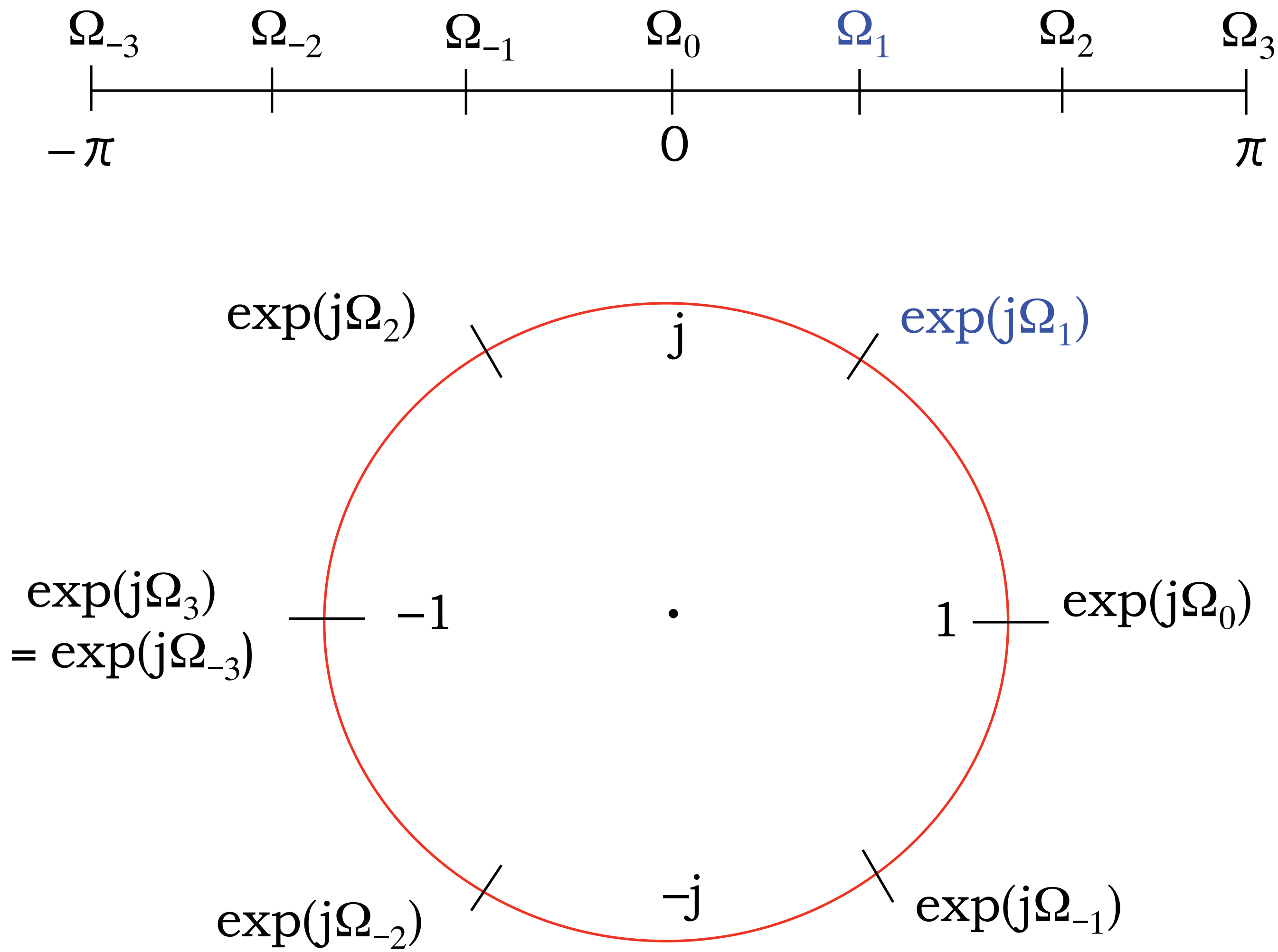
where $\Omega_k = k(2\pi/P)$, P is some integer (preferably a power of 2) such that P is longer than the time interval [0,L-1] over which x[n] is nonzero, and k ranges from –P/2 to (P/2)–1 (for even P).

Computing these series involves $O(P^2)$ operations – when P gets large, the computations get very s l o w….

Happily, in 1965 Cooley and Tukey published a fast method for computing the Fourier transform (aka **FFT**, IFFT), rediscovering a technique known to Gauss. This method takes O(P log P) operations.

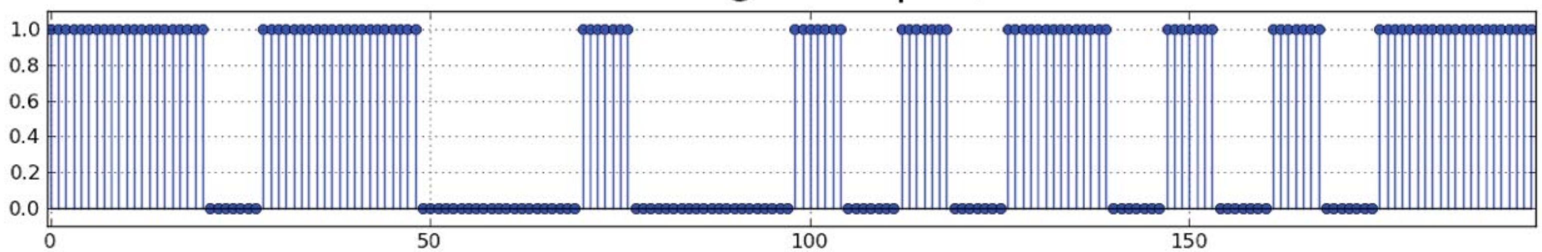
$P = 1024$, $P^2 = 1{,}048{,}576$, $P \log P \approx 10{,}240$

Where do the Ω_k live? e.g., for P=6 (even)

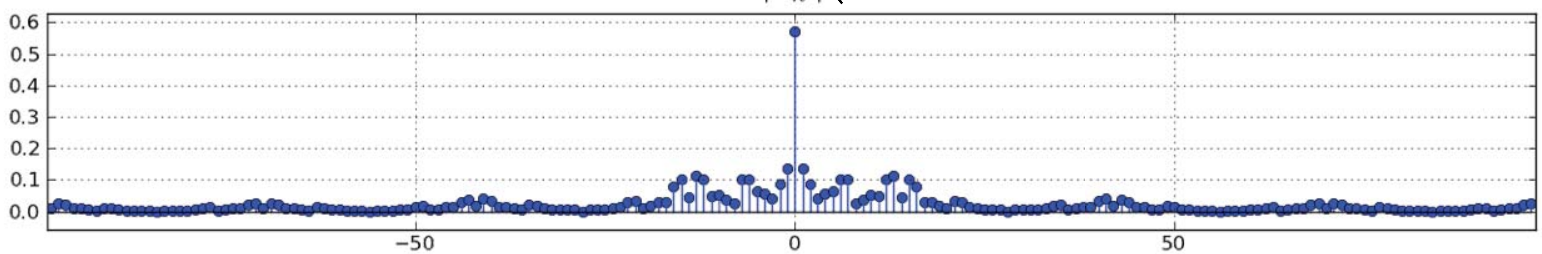


Spectrum of Digital Transmissions

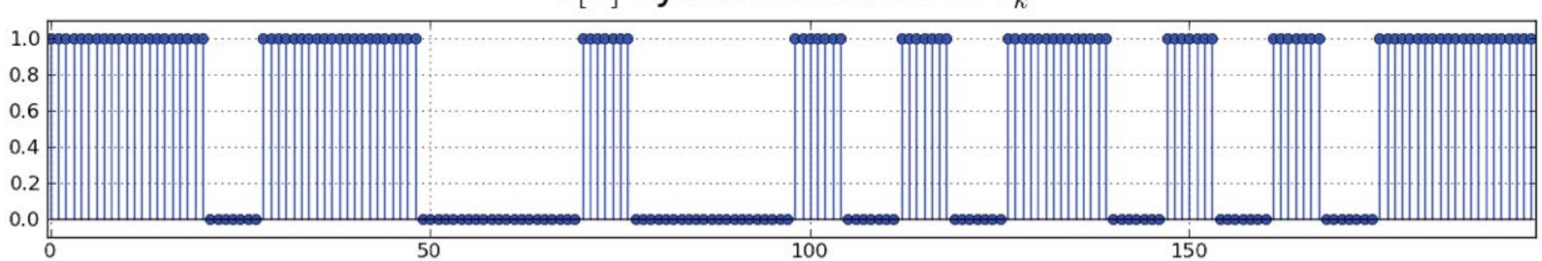
transmit @ 7 samples/bit



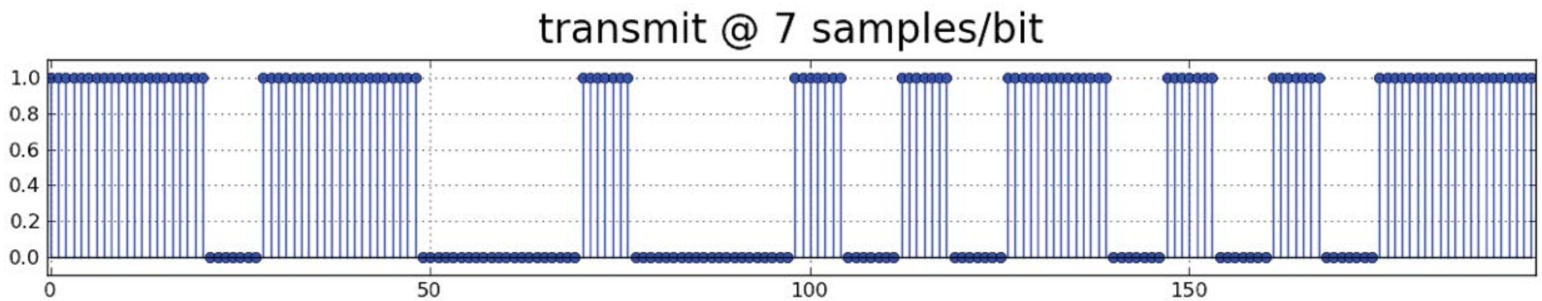
$|a_k|$(scaled version of DTFT samples)

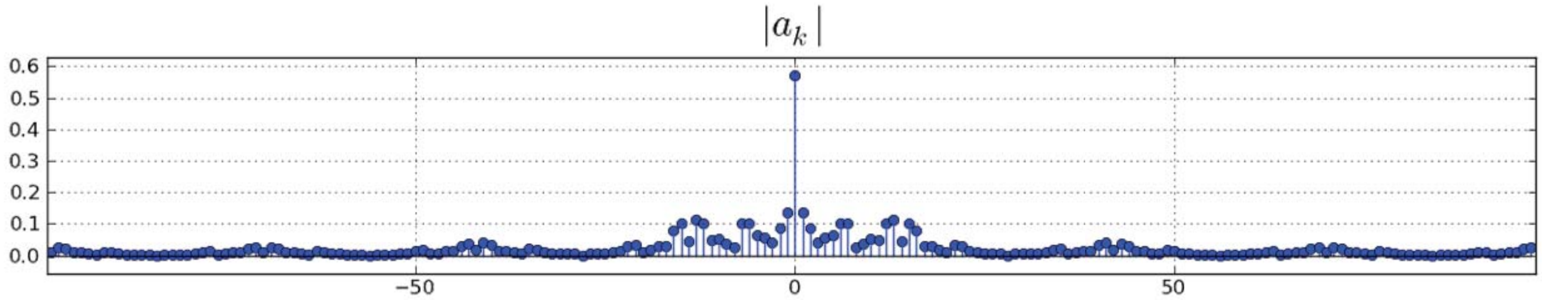


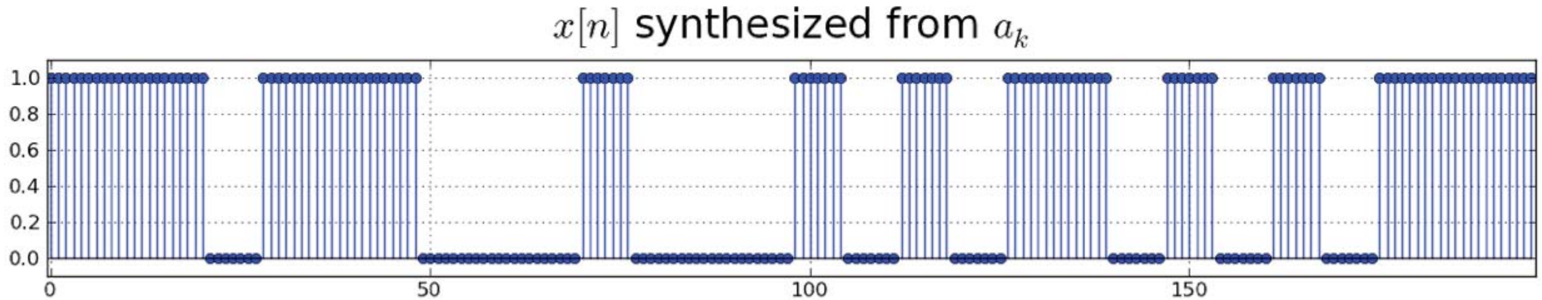
$x[n]$ synthesized from a_k



Spectrum of Digital Transmissions







Observations on previous figure

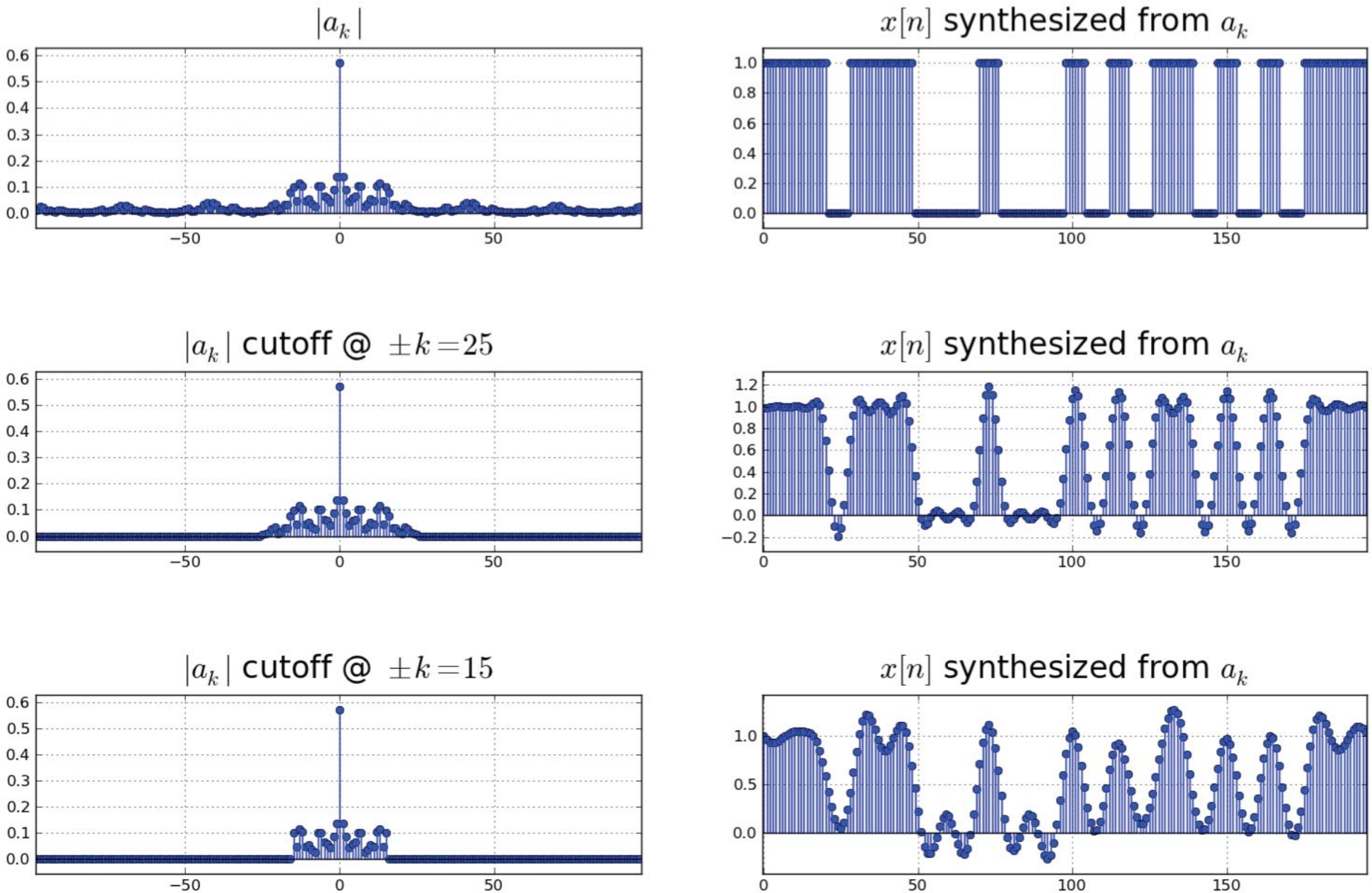
- The waveform x[n] cannot vary faster than the step change every 7 samples, so we expect the highest frequency components in the waveform to have a period around 14 samples. (The is rough and qualitative, as x[n] is not sinusoidal.)

- A period of 14 corresponds to a frequency of $2\pi/14 = \pi/7$, which is 1/7 of the way from 0 to the positive end of the frequency axis at π (so k approximately 100/7 or 14 in the figure). And that indeed is the neighborhood of where the Fourier coefficients drop off significantly in magnitude.

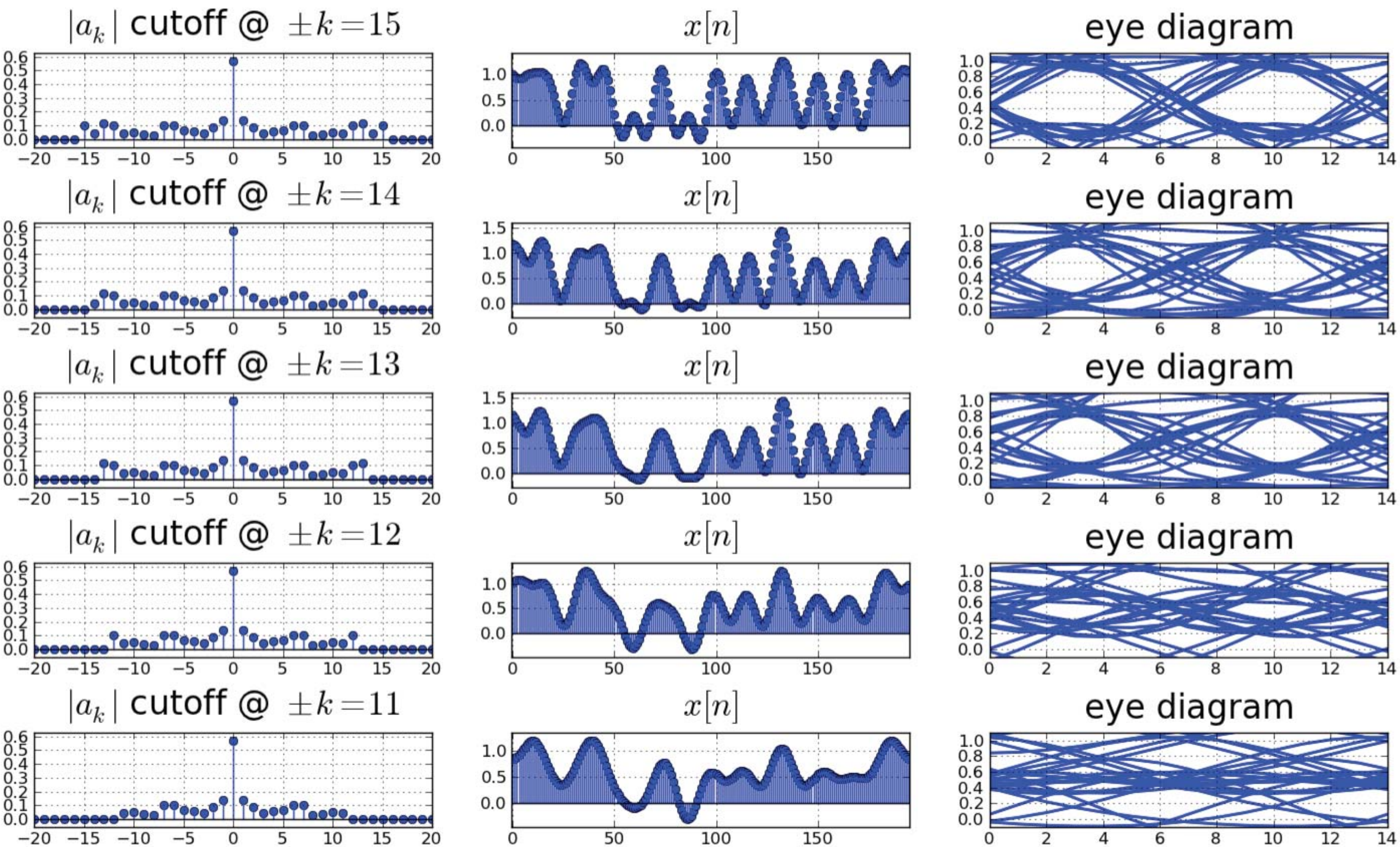
- There are also lower-frequency components corresponding to the fact that the 1 or 0 level may be held for several bit slots.

- And there are higher-frequency components that result from the transitions between voltage levels being sudden, not gradual.

Effect of Low-Pass Channel



How Low Can We Go?



7 samples/bit → 14 samples/period → k=(N/14)=(196/14)=14