Chapter 4: Static and Quasistatic Fields

4.1 Introduction

Static electric and magnetic fields are governed by the static forms of Maxwell's equations in differential and integral form for which $\partial/\partial t \rightarrow 0$:

$$\nabla \times \overline{E} = 0 \qquad \oint_C \overline{E} \bullet d\overline{s} = 0 \qquad \textit{Faraday's Law} \qquad (4.1.1)$$

$$\nabla \times \overline{H} = \overline{J} \qquad \oint_C \overline{H} \bullet d\overline{s} = \iint_A \overline{J} \bullet \hat{n}\, da \qquad \textit{Ampere's Law} \qquad (4.1.2)$$

$$\nabla \bullet \overline{D} = \rho \qquad \oiint_A (\overline{D} \bullet \hat{n})\, da = \iiint_V \rho\, dv = Q \qquad \textit{Gauss's Law} \qquad (4.1.3)$$

$$\nabla \bullet \overline{B} = 0 \qquad \oiint_A (\overline{B} \bullet \hat{n})\, da = 0 \qquad \textit{Gauss's Law} \qquad (4.1.4)$$

As shown in (1.3.5), Gauss's law (4.1.3) leads to the result that a single point charge Q at the origin in vacuum yields produces an electric field at radius r of:

$$\overline{E}(r) = \hat{r}\, Q/4\pi\varepsilon_o r^2 \qquad (4.1.5)$$

Superposition of such contributions to $\overline{E}(\overline{r})$ from a charge distribution $\rho(\overline{r}\,')$ located within the volume V' yields:

$$\overline{E}(\overline{r}) = \hat{r} \iiint_{V'} \frac{\rho(\overline{r}\,')}{4\pi\varepsilon_o |\overline{r} - \overline{r}\,'|^2} dv' \qquad \textit{Coulomb's superposition integral} \qquad (4.1.6)$$

where $\hat{r}$ is outside the integral because $r >> \sqrt[3]{V'}$. A more complex derivation given in Section 10.1 yields the corresponding equation for static magnetic fields:

$$\overline{H}(\overline{r}, t) = \iiint_{V'} \frac{\overline{J}\,' \times (\overline{r} - \overline{r}\,')}{4\pi\, |\, \overline{r} - \overline{r}\,'\,|^3} dv' \qquad \textit{Biot-Savart law} \qquad (4.1.7)$$

Any static electric field can be related to an electric potential distribution Φ[volts] because $\nabla \times \overline{E} = 0$ implies $\overline{E} = -\nabla\Phi$, where the voltage difference between two points (1.3.12) is:

$$\Phi_1 - \Phi_2 = \int_1^2 \overline{E} \bullet d\overline{s} \qquad (4.1.8)$$

Similarly, in current-free regions of space $\nabla \times \overline{H} = 0$ implies $\overline{H} = -\nabla\Psi$ [Amperes], where Ψ is magnetic potential. Therefore the magnetic potential difference between two points is:

$$\Psi_1 - \Psi_2 = \int_1^2 \overline{H} \bullet d\overline{s} \tag{4.1.9}$$

This definition of magnetic potential is useful in understanding the magnetic circuits discussed in Section 4.4.3.

Often not all source charges and currents are given because some reside on given equipotential surfaces and assume an unknown distribution consistent with that constraint. To address this case, Maxwell's equations can be simply manipulated to form Laplace's equation, which can sometimes be solved by separation of variables, as discussed in Section 4.5, or usually by numerical methods. Section 4.6 then discusses the utility of flux tubes and field mapping for understanding static field distributions.

Quasistatics assumes that the field strengths change so slowly that the electric and magnetic fields induced by those changes (the contributions to $\overline{E}$ and $\overline{H}$ from the $\partial/\partial t$ terms in Faraday's and Ampere's laws) are sufficiently small that their own induced fields ($\propto(\partial/\partial t)^2$) can be neglected; only the original and first-order induced fields are therefore of interest. Quasistatic examples were discussed in Chapter 3 in the context of resistors, capacitors, and inductors. The mirror image technique described in Section 4.2 is used for static, quasistatic, and dynamic problems and incidentally in the discussion in Section 4.3 concerning exponential relaxation of field strengths in conducting media and skin depth.

4.2 Mirror image charges and currents

One very useful problem solving technique is to change the problem definition to one that is easier to solve but is known to have the same answer. An excellent example of this approach is the use of mirror-image charges and currents, which also works for wave problems.[10]

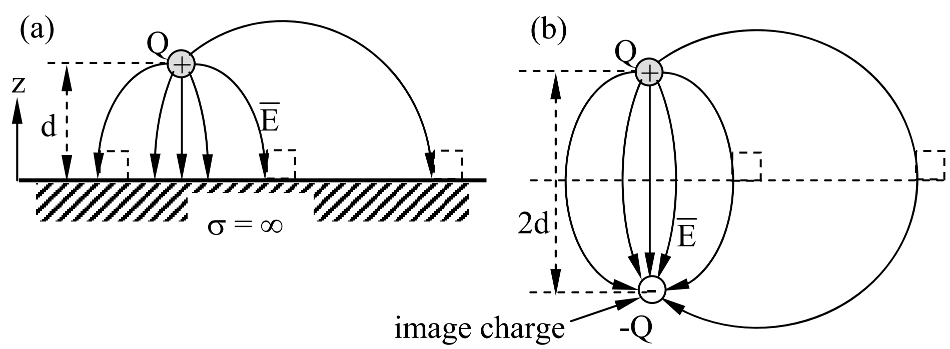

Figure 4.2.1 Image charge for an infinite planar perfect conductor.

[10] Another example of this approach is use of duality between $\overline{E}$ and $\overline{H}$, as discussed in Section 9.2.6.

Consider the problem of finding the fields produced by a charge located a distance d above an infinite perfectly conducting plane, as illustrated in Figure 4.2.1(a). Boundary conditions at the conductor require only that the electric field lines be perpendicular to its surface. Any other set of boundary conditions that imposes the same constraint must yield the same unique solution by virtue of the uniqueness theorem of Section 2.8.

One such set of equivalent boundary conditions invokes a duplicate *mirror image charge* a distance 2d away from the original charge and of opposite sign; the conductor is removed. The symmetry for equal and opposite charges requires the electric field lines $\overline{E}$ to be perpendicular to the original surface of the conductor at $z = 0$; this results in $\overline{E}$ being exactly as it was for $z > 0$ when the conductor was present, as illustrated in Figure 4.2.1(b). Therefore uniqueness says that above the half-plane the fields produced by the original charge plus its mirror image are identical to those of the original problem. The fields below the original half plane are clearly different, but they are not relevant to the original problem.

This equivalence applies for multiple charges or for a charge distribution, as illustrated in Figure 4.2.2. In fact the mirror image method remains valid so long as the charges change value or position slowly with respect to the relaxation time ε/σ of the conductor, as discussed in Section 4.4.1. The relaxation time is the 1/e time constant required for the charges within the conductor to approach new equilibrium positions after the source charge distribution outside changes.

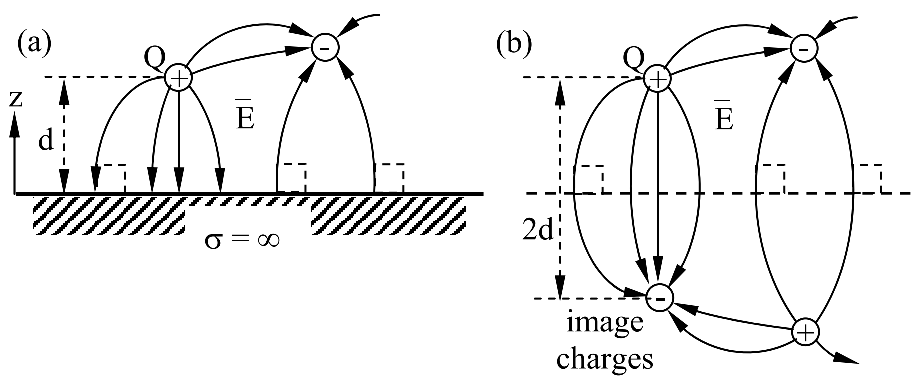

Figure 4.2.2 Multiple image charges.

Because the mirror image method works for varying or moving charges, it works for the currents that must be associated with them by conservation of charge (2.1.21), as suggested in Figure 4.2.3 (a) and (b). Figure 4.2.3(d) also suggests how the magnetic fields produced by these currents satisfy the boundary conditions for the conducting plane: at the surface of a perfect conductor $\overline{H}$ is only parallel.

The mirror image method continues to work if the upper half plane contains a conductor, as illustrated in Figure 4.2.4; the conductor must be imaged too. These conductors can even be at angles, as suggested in Figure 4.2.4(b). The region over which the deduced fields are valid is naturally restricted to the original opening between the conductors. Still more complex image configurations can be used for other conductor placements, and may even involve an infinite series of progressively smaller image charges and currents.

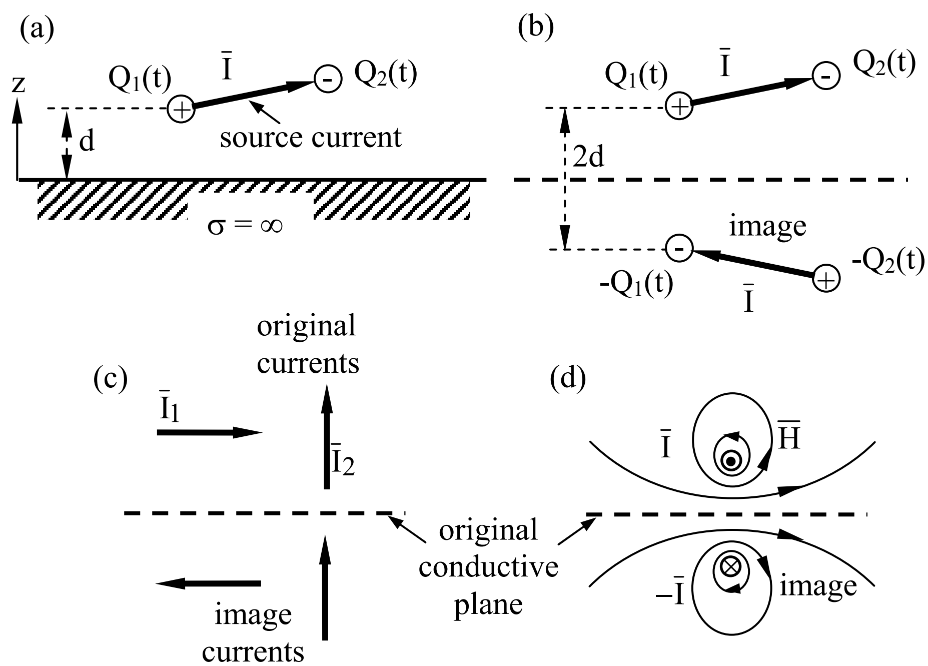

Figure 4.2.3 Image currents.

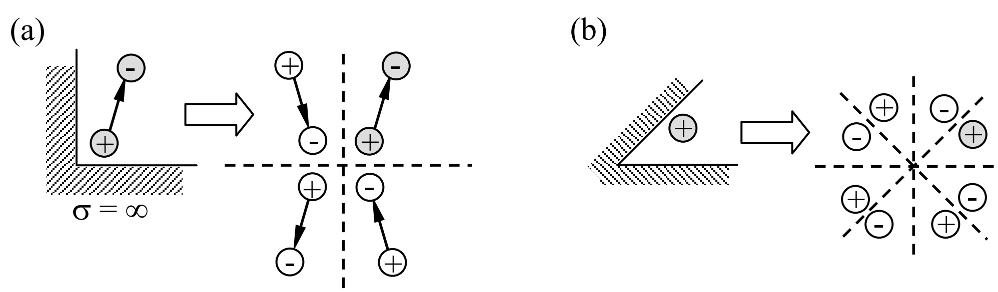

Figure 4.2.4 Image charges and currents for intersecting conductors.

4.3 *Relaxation of fields, skin depth*

4.3.1 Relaxation of electric fields and charge in conducting media

Electric and magnetic fields established in conducting time-invariant homogeneous media tend to decay exponentially unless maintained. Under the quasistatic assumption all time variations are sufficiently slow that contributions to $\overline{E}$ by $\partial\overline{B}/\partial t$ are negligible, which avoids wave-like behavior and simplifies the problem. This relaxation process is governed by the conservation-of-charge relation (2.1.21), Gauss's law $\left(\nabla \bullet \overline{D} = \rho\right)$, and Ohm's law $\left(\overline{J} = \sigma\overline{E}\right)$:

$$\nabla \bullet \bar{J} + \partial\rho/\partial t = 0 = \nabla \bullet (\sigma\bar{E}) + (\partial/\partial t)(\nabla \bullet \varepsilon\bar{E}) = \nabla \bullet [(\sigma + \varepsilon\partial/\partial t)\bar{E}] = 0 \quad (4.3.1)$$

Since an arbitrary $\bar{E}$ can be established by initial conditions, the general solution to (4.3.1) requires $(\sigma + \varepsilon\partial/\partial t)\nabla \bullet \bar{E} = 0$, leading to the differential equation:

$$(\partial/\partial t + \sigma/\varepsilon)\rho = 0 \quad (4.3.2)$$

where $\nabla \bullet \bar{E} = \rho/\varepsilon$. This has the solution that $\rho(\bar{r})$ relaxes exponentially with a charge *relaxation time* constant $\tau = \varepsilon/\sigma$ seconds:

$$\rho(\bar{r}) = \rho_o(\bar{r})e^{-\sigma t/\varepsilon} = \rho_o(\bar{r})e^{-t/\tau}\left[\mathrm{m}^{-3}\right] \quad (\textit{charge relaxation}) \quad (4.3.3)$$

It follows that an arbitrary initial electric field $\bar{E}(\bar{r})$ in a medium having uniform ε and σ will also decay exponentially with the same time constant ε/σ because Gauss's law relates $\bar{E}$ and ρ linearly:

$$\nabla \bullet \bar{E} = \rho(t)/\varepsilon \quad (4.3.4)$$

where $\nabla \bullet \bar{E}_o \equiv \rho_o/\varepsilon$. Therefore *electric field relaxation* is characterized by:

$$\bar{E}(\bar{r}, t) = \bar{E}_o(\bar{r})e^{-\sigma t/\varepsilon}\left[\mathrm{v\ m}^{-1}\right] \quad (\text{electric field relaxation}) \quad (4.3.5)$$

We should expect such exponential decay because any electric fields in a conductor will generate currents and therefore dissipate power proportional to J^2 and E^2. But the stored electrical energy is also proportional to E^2, and power dissipation is the negative derivative of stored energy. That is, the energy decays at a rate proportional to its present value, which results in exponential decay. In copper $\tau = \varepsilon_o/\sigma \cong 9\times10^{-12}/(5\times10^{7}) \cong 2\times10^{-19}$ seconds, short compared to any delay of common interest. The special case of parallel-plate resistors and capacitors is discussed in Section 3.1.

Example 4.3A

What are the electric field relaxation time constants τ for sea water ($\varepsilon \cong 80\varepsilon_o$, $\sigma \cong 4$) and dry soil ($\varepsilon \cong 2\varepsilon_o$, $\sigma \cong 10^{-5}$)? For what radio frequencies can they be considered good conductors?

Solution: Equation (4.3.5) yields $\tau = \varepsilon/\sigma \cong (80\times8.8\times10^{-12})/4 \cong 1.8\times10^{-10}$ seconds for seawater, and $(2\times8.8\times10^{-12})/10^{-5} \cong 1.8\times10^{-6}$ seconds for dry soil. So long as $\bar{E}$ changes slowly with respect to τ, the medium has time to cancel $\bar{E}$; frequencies below ~5 GHz and ~500 kHz have this property for seawater and typical dry soil, respectively, which behave like good conductors at these lower frequencies. Moist soil behaves like a conductor up to ~5 MHz and higher.

4.3.2 Relaxation of magnetic fields in conducting media

Magnetic fields and their induced currents similarly decay exponentially in conducting media unless they are externally maintained; this decay process is often called *magnetic diffusion* or *magnetic relaxation*. We assume that the time variations are sufficiently slow that contributions to $\overline{H}$ by $\partial\overline{D}/\partial t$ are negligible. In this limit Ampere's law becomes:

$$\nabla\times\overline{H} = \overline{J} = \sigma\overline{E} \tag{4.3.6}$$

$$\nabla\times(\nabla\times\overline{H}) = \sigma\nabla\times\overline{E} = -\sigma\mu\partial\overline{H}/\partial t = -\nabla^2\overline{H} + \nabla(\nabla\bullet\overline{H}) = -\nabla^2\overline{H} \tag{4.3.7}$$

where Faraday's law, the vector identity (2.2.6), and Gauss's law $(\nabla\bullet\overline{B} = 0)$ were used.

The resulting differential equation:

$$\sigma\mu\partial\overline{H}/\partial t = \nabla^2\overline{H} \tag{4.3.8}$$

has at least one simple solution:

$$\overline{H}(z,t) = \hat{x}H_o e^{-t/\tau_m}\cos kz \tag{4.3.9}$$

where we assumed an x-polarized z-varying sinusoid. Substituting (4.3.9) into (4.3.8) yields the desired time constant:

$$\tau_m = \mu\sigma/k^2 = \mu\sigma\lambda^2/4\pi^2 \quad [s] \qquad \text{(magnetic relaxation time)} \tag{4.3.10}$$

Thus the lifetime of magnetic field distributions in conducting media increases with permeability (energy storage density), conductivity (reducing dissipation for a given current), and the wavelength squared ($\lambda = 2\pi/k$).

4.3.3 Induced currents

Quasistatic magnetic fields induce electric fields by virtue of Faraday's law: $\nabla\times\overline{E} = -\mu\partial\overline{H}/\partial t$. In conductors these induced electric fields drive currents that obey *Lenz's law*: "The direction of induced currents tends to oppose changes in magnetic flux." Induced currents find wide application, for example, in: 1) heating, as in induction furnaces that melt metals, 2) mechanical actuation, as in induction motors and impulse generators, and 3) electromagnetic shielding. In some cases these induced currents are undesirable and are inhibited by subdividing the conductors into elements separated by thin insulating barriers. All these examples are discussed below.

First consider a simple conducting hollow cylinder of length W driven circumferentially by current $I_o u(t)$, as illustrated in Figure 4.3.1, where u(t) is the unit step function (the current is zero until t = 0, when it becomes I_o). Centered in the outer cylinder is an isolated second cylinder of conductivity σ and having a thin wall of thickness δ; its length and diameter are W and D << W, respectively.

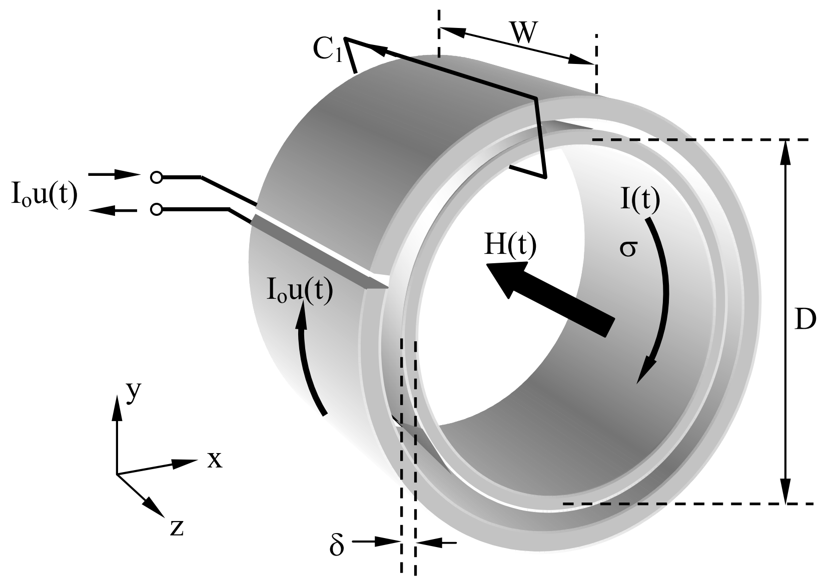

Figure 4.3.1 Relaxation penetration of a magnetic field into a conducting cylinder.

If the inner cylinder were a perfect conductor, then the current $I_o u(t)$ would produce an equal and opposite image current $\sim -I_o u(t)$ on the outer surface of the inner cylinder, thus producing a net zero magnetic field inside the cylinder formed by that image current. Consider the integral of $\overline{H} \bullet d\overline{s}$ around a closed contour C_1 that threads both cylinders and circles zero net current at t = 0+; this integral yields zero. If the inner conductor were slightly resistive, then the same equal and opposite current would flow on the inner cylinder, but it would slowly dissipate heat until the image current decayed to zero and the magnetic field inside reached the maximum value I_o/W [A m^{-1}] associated with the outer current I_o. These conclusions are quantified below.

The magnetic field H inside the inner cylinder depends on the currents flowing in the outer and inner cylinders, I_o and I(t), respectively:

$$H(t) = u(t)[I_o + I(t)]/W \qquad (4.3.11)$$

The current I(t) flowing in the inner cylinder is driven by the voltage induced by H(t) via Faraday's law (2.4.14):

$$\oint_{C_2} \overline{E} \bullet d\overline{s} = IR = \mu_o \int_A \left(d\overline{H}/dt\right) \bullet d\overline{a} = \mu_o A\, dH/dt \qquad (4.3.12)$$

where the contour C_2 is in the x-y plane and circles the inner cylinder with diameter D. The area circled by the contour $A = \pi D^2/4$. The circumferential resistance of the inner cylinder is R =

$\pi D/\sigma\delta W$ ohms. For simplicity we assume that the permeability here is μ_o everywhere. Substituting (4.3.11) into (4.3.12) yields a differential equation for I(t):

$$I(t) = -(\mu_o A/WR)\, dI/dt \qquad (4.3.13)$$

Substituting the general solution $I(t) = Ke^{-t/\tau}$ into (4.3.13) yields:

$$Ke^{-t/\tau} = (\mu_o A/WR\tau)Ke^{-t/\tau} \qquad (4.3.14)$$

$$\tau = \mu_o A/WR = \mu_o A\sigma\delta/\pi D \ \ [s] \qquad \text{(magnetic relaxation time)} \qquad (4.3.15)$$

Thus the greater the conductivity of the inner cylinder, and the larger its product μA, the longer it takes for transient magnetic fields to penetrate it. For the special case where $\delta = D/4\pi$ and $A = D^2$, we find $\tau = \mu_o\sigma(D/2\pi)^2$, which is the same magnetic relaxation time constant derived in (4.3.10) if we identify D with the wavelength λ of the magnetic field variations. Equation (4.3.15) is also approximately correct if $\mu_o \rightarrow \mu$ for the inner cylinder.

Since $H(t) = 0$ at $t = 0+$, (4.3.11) yields $I(t = 0+) = -I_o$, and the solution $I(t) = Ke^{-t/\tau}$ becomes:

$$I(t) = -I_o\, e^{-t/\tau} \ \ [A] \qquad (4.3.16)$$

The magnetic field inside the inner cylinder follows from (4.3.16) and (4.3.11):

$$H(t) = u(t)I_o\,(1 - e^{-t/\tau})/W \ \ [A\ m^{-1}] \qquad (4.3.17)$$

The geometry of Figure 4.3.1 can be used to heat resistive materials such as metals electrically by placing the metals in a ceramic container that sinusoidal magnetic fields penetrate easily. The induced currents can then melt the material quicker by heating the material throughout rather than just at the surface, as would a flame. The frequency f generally must be sufficiently low that the magnetic fields penetrate a significant fraction of the container diameter; $f << 1/\tau$.

The inner cylinder of Figure 4.3.1 can also be used to shield its interior from alternating magnetic fields by designing it so that its time constant τ is much greater than the period of the undesired AC signal; large values of $\mu\sigma\delta$ facilitate this since $\tau = \mu_o A\sigma\delta/\pi D$ (4.3.15). Since we can model a solid inner cylinder as a continuum of concentric thin conducting shells, it follows that the inner shells will begin to see significant magnetic fields only after the surrounding shells do, and therefore the time delay experienced increases with depth. This is consistent with $\tau \propto \delta$. The penetration of alternating fields into conducting surfaces is discussed further in Section 9.3 in terms of the exponential penetration skin depth $\delta = \sqrt{2/\omega\mu\sigma}$ [m].

Two actuator configurations are suggested by Figure 4.3.1. First, the inner cylinder could be inserted only part way into the outer cylinder. Then the net force on the inner cylinder would expel it when the outer cylinder was energized because the polarity of these two electromagnets are reversed, the outer one powered by I_o and the inner one by $-I_o(1 - e^{-t/\tau})$. Electromagnetic forces are discussed more fully in Chapter 5; here it suffices to note that induced currents can be used to simplify electromechanical actuators. A similar "kick" can be applied to a flat plate placed across the end of the outer cylinder, for again the induced cylindrically shaped mirror image current would experience a transient repulsive force. Mirror-image currents were discussed in Section 4.2.

The inner cores in transformers and some inductors are typically iron and are circled by wires carrying alternating currents, as discussed in Section 3.2. The alternating currents induce circular currents in the core called *eddy currents* that dissipate power. To minimize such induced currents and losses, high-μ conducting cores are commonly composed of many thin sheets separated from each other by thin coats of varnish or other insulator that largely blocks those induced currents; these are called *laminated cores*. A rough estimate of the effectiveness of using N plates instead of one can be obtained by noting that the power P_d dissipated in each lamination is proportional to V^2/R, where $V = \oint_C \overline{E} \bullet d\overline{s}$ is the loop voltage induced by H(t) and R is the effective resistance of that loop. By design H(t) usually penetrates the full transformer core. Thus V is roughly proportional to the area of each lamination in the plane perpendicular to $\overline{H}$, which decreases as 1/N. The resistance R experienced by the induced current circulating in each lamination increases roughly by N since the width of the channel through which it can flow is reduced as N increases while the length of the channel changes only moderately. The total power dissipated for N laminations is thus roughly proportional to $NV^2/R \propto NN^{-2}/N = N^{-2}$. Therefore we need only increase N to the point where the power loss is tolerable and the penetration of the transformer core by H(t) is nearly complete each period.

Example 4.3B

How long does it take a magnetic field to penetrate a 1-mm thick metal cylinder of diameter D with conductivity 5×10^7 [S/m] if $\mu = \mu_o$? Design a shield for a ~10-cm computer that blocks 1-MHz magnetic fields emanating from an AM radio.

Solution: If we assume the geometry of Figure 4.3.1 and use (4.3.15), $\tau = \mu_o A\sigma\delta/\pi D$, we find $\tau = 1.3\times10^{-6}\times D\times5\times10^7\times10^{-3}/4 = 0.016D$ seconds, where $A = \pi D^2/4$ and $\delta = 10^{-3}$. If $D = 0.1$, then $\tau = 1.6\times10^{-2}$ seconds, which is $\sim10^5$ longer than the rise time $\sim10^{-6}/2\pi$ of a 1-MHz signal. If a smaller ratio of 10^2 is sufficient, then a one-micron thick layer of metal evaporated on thin plastic might suffice. If the metal had $\mu = 10^4\mu_o$, then a one-micron thick layer would provide a safety factor of 10^6.

4.4 Static fields in inhomogeneous materials

4.4.1 Static electric fields in inhomogeneous materials

Many practical problems involve *inhomogeneous media* where the boundaries may be abrupt, as in most capacitors or motors, or graded, as in many semiconductor or optoelectronic devices. The basic issues are well illustrated by the static cases discussed below. Sections 4.4.1 and 4.4.2 discuss static electric and magnetic fields, respectively, in inhomogeneous media. To simplify the discussion, only media characterized by real scalar values for ε, μ, and σ will be considered, where all three properties can be a function of position.

Static electric fields in all media are governed by the static forms of Faraday's and Gauss's laws:

$$\nabla \times \overline{E} = 0 \tag{4.4.1}$$

$$\nabla \bullet \overline{D} = \rho_f \tag{4.4.2}$$

and by the constitutive relations:

$$\overline{D} = \varepsilon\overline{E} = \varepsilon_o \overline{E} + \overline{P} \tag{4.4.3}$$

$$\overline{J} = \sigma\overline{E} \tag{4.4.4}$$

A few simple cases illustrate how these laws can be used to characterize inhomogeneous conductors and dielectrics. Perhaps the simplest case is that of a wire or other conducting structure (1) imbedded in a perfectly insulating medium (2) having conductivity σ = 0. Since charge is conserved, the perpendicular components of current must be the same on both sides of the boundary so that $J_{1\perp} = J_{2\perp} = 0 = E_{2\perp}$. Therefore all currents in the conducting medium are trapped within it and at the surface must flow parallel to that surface.

Let's consider next the simple case of an inhomogeneous slab between two parallel perfectly conducting plates spaced L apart in the x direction at a potential difference of V_o volts, where the terminal at x = 0 has the greater voltage. Suppose that the medium has permittivity ε, current density J_o, and inhomogeneous conductivity σ(x), where:

$$\sigma = \sigma_o \Big/ \left[1 + \frac{x}{L}\right] \ \left[\text{Siemens m}^{-1}\right] \tag{4.4.5}$$

The associated electric field follows from (4.4.4):

$$\overline{E} = \overline{J}/\sigma = \hat{x}\frac{J_o}{\sigma_o}\left(1 + \frac{x}{L}\right) \ \left[\text{Vm}^{-1}\right] \tag{4.4.6}$$

The free charge density in the medium then follows from (4.4.2) and is:

$$\rho_f = \nabla \bullet \overline{D} = \left(\varepsilon J_o/\sigma_o\right)\left(\partial/\partial x\right)\left(1 + x/L\right) = \varepsilon J_o/\sigma_o L \ \left[\text{Cm}^{-3}\right] \tag{4.4.7}$$

Note from the derivative in (4.4.7) that abrupt discontinuities in conductivity generally produce free surface charge ρ_s at the discontinuity. Although inhomogeneous conductors have a net free charge density throughout the volume, they may or may not also have a net polarization charge density $\rho_p = -\nabla \bullet \overline{P}$, which is defined in (2.5.12) and can be deduced from the polarization vector $\overline{P} = \overline{D} - \varepsilon_o \overline{E} = (\varepsilon - \varepsilon_o)\overline{E}$ using (4.4.7):

$$\rho_p = -\nabla \bullet \overline{P} = -\nabla \bullet \left[(\varepsilon - \varepsilon_o)\overline{E}\right] = (\varepsilon - \varepsilon_o) J_o / \sigma_o L \ \left[\text{Cm}^{-3}\right] \tag{4.4.8}$$

Now let's consider the effects of inhomogeneous permittivity ε(x) in an insulating medium ($\sigma = 0$) where:

$$\varepsilon = \varepsilon_o \left(1 + \frac{x}{L}\right) \tag{4.4.9}$$

Since the insulating slab should contain no free charge and the boundaries force $\overline{D}$ to be in the x direction, therefore $\overline{D}$ cannot be a function of x because $\nabla \bullet \overline{D} = \rho_f = 0$. But $\overline{D} = \varepsilon(x)\overline{E}(x)$; therefore the x dependence of $\overline{E}$ must cancel that of ε, so:

$$\overline{E} = \hat{x} E_o \Big/ \left(1 + \frac{x}{L}\right) \tag{4.4.10}$$

E_o is an unknown constant and can be found relative to the applied voltage V_o:

$$V_o = \int_0^L E_x \ dx = \int_0^L \left[E_o \Big/ \left(1 + \frac{x}{L}\right)\right] dx = L \ E_o \ln 2 \tag{4.4.11}$$

Combining (4.4.9–11) leads to a displacement vector $\overline{D}$ that is independent of x (boundary conditions mandate continuity of $\overline{D}$), and a non-zero polarization charge density ρ_p distributed throughout the medium:

$$\overline{D} = \varepsilon \overline{E} = \hat{x} \varepsilon_o V_o / (L \ln 2) \tag{4.4.12}$$

$$\begin{aligned} \rho_p &= -\nabla \bullet \overline{P} = -\nabla \bullet \left(\overline{D} - \varepsilon_o \overline{E}\right) = \varepsilon_o \nabla \bullet \overline{E} \\ &= \frac{\varepsilon_o V_o}{L \ln 2} \frac{\partial}{\partial x} (1 + x/L)^{-1} = \frac{-\varepsilon_o V_o}{(L + x)^2 \ln 2} \ \left[\text{Cm}^{-3}\right] \end{aligned} \tag{4.4.13}$$

A similar series of computations readily handles the case where both ε and σ are inhomogeneous.

Example 4.4A

A certain capacitor consists of two parallel conducting plates, one at z = 0 and +V volts and one at z = d and zero volts. They are separated by a dielectric slab of permittivity ε, for which the conductivity is small and different in the two halves of the dielectric, each of which is d/2 thick; $\sigma_1 = 3\sigma_2$. Assume the interface between σ_1 and σ_2 is parallel to the capacitor plates and is located at z = 0. What is the free charge density $\rho_f(z)$ in the dielectric, and what is $\overline{E}(z)$ where z is the coordinate perpendicular to the plates?

Solution: Since charge is conserved, $\overline{J}_1 = \overline{J}_2 = \sigma_1 \overline{E}_1 = \sigma_2 \overline{E}_2$, so $\overline{E}_2 = \sigma_1 \overline{E}_1 / \sigma_2 = 3\overline{E}_1$. But $(E_1+E_2)d/2 = V$, so $4E_1 d/2 = V$, and $E_1 = V/2d$. The surface charge on the lower plate is $\rho_{s(z=0)} = \overline{D}_{z=0} = \varepsilon E_1 = \varepsilon V/2d$ [C/m^2], and ρ_s on the upper plate is $-\overline{D}_{z=d} = -\varepsilon E_2 = -\varepsilon 3V/2d$. The free charge at the dielectric interface is $\rho_{s\,(z = d/2)} = D_2 - D_1 = \varepsilon(E_2 - E_1) = \varepsilon V/d$. Charge can accumulate at all three surfaces because the dielectric conducts. The net charge is zero. The electric field between capacitor plates was discussed in Section 3.1.2.

4.4.2 Static magnetic fields in inhomogeneous materials

Static magnetic fields in most media are governed by the static forms of Ampere's and Gauss's laws:

$$\nabla \times \overline{H} = 0 \tag{4.4.14}$$

$$\nabla \bullet \overline{B} = 0 \tag{4.4.15}$$

and by the constitutive relations:

$$\overline{B} = \mu \overline{H} = \mu_o \left(\overline{H} + \overline{M}\right) \tag{4.4.16}$$

One simple case illustrates how these laws characterize inhomogeneous magnetic materials. Consider a magnetic material that is characterized by $\mu(x)$ and has an imposed magnetic field $\overline{B}$ in the x direction. Since $\nabla \bullet \overline{B} = 0$ it follows that $\overline{B}$ is constant $(\overline{B}_o)$ throughout, and that $\overline{H}$ is a function of x:

$$\overline{H} = \frac{\overline{B}_o}{\mu(x)} \tag{4.4.17}$$

As a result, higher-permeability regions of magnetic materials generally host weaker magnetic fields $\overline{H}$, as shown in Section 3.2.2 for the toroidal inductors with gaps. In many magnetic devices μ might vary four to six orders of magnitude, as would $\overline{H}$.

4.4.3 Electric and magnetic flux trapping in inhomogeneous systems

Currents generally flow in conductors that control the spatial distribution of $\overline{J}$ and electric potential $\Phi(\overline{r})$. Similarly, high-permeability materials with $\mu \gg \mu_o$ can be used to form *magnetic circuits* that guide $\overline{B}$ and control the spatial form of the static curl-free *magnetic potential* $\Psi(\overline{r})$.

Faraday's law says that static electric fields $\overline{E}$ are curl-free:

$$\nabla \times \overline{E} = -\frac{\partial \overline{B}}{\partial t} = 0 \qquad \text{(Faraday's law)} \qquad (4.4.18)$$

Since $\nabla \times \overline{E} = 0$ in static cases, it follows that:

$$\overline{E} = -\nabla\Phi \qquad (4.4.19)$$

where Φ is the *electric potential* [volts] as a function of position in space. But Gauss's law says $\nabla \bullet \overline{E} = \rho/\varepsilon$ in regions where ρ is constant. Therefore $\nabla \bullet \overline{E} = -\nabla^2\Phi = \rho/\varepsilon$ and:

$$\nabla^2\Phi = -\frac{\rho}{\varepsilon} \qquad \text{(Laplace's equation)} \qquad (4.4.20)$$

In static current-free regions of space with constant permeability μ, Ampere's law (2.1.6) says:

$$\nabla \times \overline{H} = 0 \qquad (4.4.21)$$

and therefore $\overline{H}$, like $\overline{E}$, can be related to a scalar *magnetic potential* [Amperes] Ψ:

$$\overline{H} = -\nabla\Psi \qquad (4.4.22)$$

Since $\nabla \bullet \overline{H} = 0$ when μ is independent of position, it follows that $\nabla \bullet (-\nabla\Psi) = \nabla^2\Psi$ and:

$$\nabla^2\Psi = 0 \qquad \text{(Laplace's equation for magnetic potential)} \qquad (4.4.23)$$

The perfect parallel between Laplace's equations (4.4.20) and (4.4.23) for electric and magnetic fields in charge-free regions offers a parallel between current density $\overline{J} = \sigma\overline{E}$ [A/m^2] and magnetic flux density $\overline{B} = \mu\overline{H}$, and also between conductivity σ and permeability μ as they relate to gradients of electric and magnetic potential, respectively:

$$\nabla^2\Phi = 0 \qquad \nabla^2\Psi = 0 \qquad (4.4.24)$$

$$\overline{E} = -\nabla\Phi \qquad \overline{H} = -\nabla\Psi \qquad (4.4.25)$$

$$\overline{J} = \sigma\overline{E} = -\sigma\nabla\Phi \qquad \overline{B} = \mu\overline{H} = -\mu\nabla\Psi \qquad (4.4.26)$$

Just as current is confined to flow within wires imbedded in insulating media having $\sigma \cong 0$, so is magnetic flux $\overline{B}$ trapped within high-permeability materials imbedded in very low permeability media, as suggested by the discussion in Section 3.2.2 of how magnetic fields are confined within high-permeability toroids.

The boundary condition (2.6.5) that $\overline{B}_\perp$ is continuous requires that $\overline{B}_\perp \cong 0$ at boundaries with media having $\mu \cong 0$; thus essentially all magnetic flux $\overline{B}$ is confined within permeable magnetic media having $\mu >> 0$.

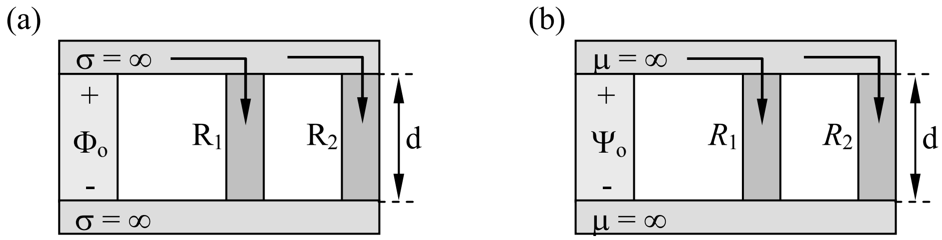

Figure 4.4.1 Current and magnetic flux-divider circuits.

Two parallel examples that help clarify the issues are illustrated in Figure 4.4.1. In Figure 4.4.1(a) a battery connected to perfect conductors apply the same voltage Φ_o across two conductors in parallel; A_i, σ_i, d_i, and I_i are respectively their cross-sectional area, conductivity, length, and current flow for i = 1,2. The current through each conductor is given by (4.4.26) and:

$$I_i = J_iA_i = \sigma_\iota\nabla\Phi_iA = \sigma_\iota\Phi_oA_i/d_i = \Phi_o/R_i \tag{4.4.27}$$

where:

$$R_i = d_i/\sigma_iA_i \ \ [\text{ohms}] \tag{4.4.28}$$

is the *resistance* of conductor i, and I = V/R is *Ohm's law*.

For the *magnetic circuit* of Figure 4.4.1(b) a parallel set of relations is obtained, where the total magnetic flux $\Lambda = BA$ [*Webers*] through a cross-section of area A is analogous to current I = JA. The magnetic flux Λ through each magnetic branch is given by (4.4.26) so that:

$$\Lambda_i = B_iA_i = \mu_\iota\nabla\Psi_iA_i = \mu_\iota\Psi_oA_i/d_i = \Psi_o/\mathcal{R}_i \tag{4.4.29}$$

where:

$$\mathcal{R}_i = d_i/\mu_iA \tag{4.4.30}$$

is the magnetic *reluctance* of branch i, analogous to the resistance of a conductive branch.

Because of the parallel between current I and magnetic flux Λ, they divide similarly between alternative parallel paths. That is, the total current is:

$$I_o = I_1 + I_2 = \Phi_0(R_1 + R_2)/R_1R_2 \tag{4.4.31}$$

The value of Φ_o found from (4.4.31) leads directly to the *current-divider equation*:

$$I_1 = \Phi_o/R_1 = I_o R_2/(R_1 + R_2) \tag{4.4.32}$$

So, if $R_2 = \infty$, all I_o flows through R_1; $R_2 = 0$ implies no current flows through R_1; and $R_2 = R_1$ implies half flows through each branch. The corresponding equations for total magnetic flux and flux division in magnetic circuits are:

$$\Lambda_o = \Lambda_1 + \Lambda_2 = \Psi_0(\mathcal{R}_1 + \mathcal{R}_2)/\mathcal{R}_1\mathcal{R}_2 \tag{4.4.33}$$

$$\Lambda_1 = \Psi_o/\mathcal{R}_1 = \Lambda_o\mathcal{R}_2/(\mathcal{R}_1 + \mathcal{R}_2) \tag{4.4.34}$$

Although the conductivity of insulators surrounding wires is generally over ten orders of magnitude smaller than that of the wires, the same is not true for the permeability surrounding high-μ materials, so there generally is some small amount of flux leakage from such media; the trapping is not perfect. In this case $\overline{H}$ outside the high-μ material is nearly perpendicular to its surface, as shown in (2.6.13).

Example 4.4B

The magnetic circuit of Figure 4.4.1(b) is driven by a wire that carries 3 amperes and is wrapped 50 times around the leftmost vertical member in a clockwise direction as seen from the top. That member has infinite permeability ($\mu = \infty$), as do the top and bottom members. If the rightmost member is missing, what is the magnetic field $\overline{H}$ in the vertical member R_1, for which the length is d and $\mu >> \mu_o$? If both R_1 and R_2 are in place and identical, what then are $\overline{H}_1$ and $\overline{H}_2$? If R_2 is removed and R_1 consists of two long thin bars in series having lengths d_a and d_b, cross-sectional areas A_a and A_b, and permeabilities μ_a and μ_b, respectively, what then are $\overline{H}_a$ and $\overline{H}_b$?

Solution: For this static problem Ampere's law (4.1.2) becomes $\oint_C \overline{H} \bullet d\overline{s} = \oiint_A \overline{J} \bullet \hat{n}\, da = N\, I = 50\times3 = 150$ [A] $= Hd$. Therefore $\overline{H} = \hat{z}\,150/d$ [A m^{-1}], where $\hat{z}$ and $\overline{H}$ are upward due to the right-hand rule associated with Ampere's law. If R_2 is added, both the integrals of $\overline{H}$ through the two branches must still equal NI, so $\overline{H}$ remains $\hat{z}\,150/d$ [A m^{-1}] in both branches. For the series case the integral of $\overline{H}$ yields $H_a d_a + H_b d_b = NI$. Because the magnetic flux is trapped within this branch, it is constant: $\mu_a H_a A_a = B_a A_a = B_b A_b = \mu_b H_b A_b$. Therefore $H_b = H_a(\mu_a A_a/\mu_b A_b)$ and $H_a[d_a + d_b(\mu_a A_a/\mu_b A_b)] = NI$, so $\overline{H}_a = \hat{z}\,NI/[d_a + d_b(\mu_a A_a/\mu_b A_b)]$ [A m^{-1}].

4.5 Laplace's equation and separation of variables

4.5.1 Laplace's equation

Electric and magnetic fields obey Faraday's and Ampere's laws, respectively, and when the fields are static and the charge and current are zero we have:

$$\nabla \times \overline{E} = 0 \qquad (4.5.1)$$

$$\nabla \times \overline{H} = 0 \qquad (4.5.2)$$

These equations are satisfied by any $\overline{E}$ or $\overline{H}$ that can be expressed as the gradient of a potential:

$$\overline{E} = -\nabla\Phi \qquad (4.5.3)$$

$$\overline{H} = -\nabla\Psi \qquad (4.5.4)$$

Therefore Maxwell's equations for static charge-free regions of space are satisfied for any arbitrary differentiable potential function $\Phi(\overline{r})$ or $\Psi(\overline{r})$, which can be determined as discussed below.

Any potential function must be consistent with the given boundary conditions, and with Gauss's laws in static charge- and current-free spaces:

$$\nabla \bullet \overline{D} = 0 \qquad (4.5.5)$$

$$\nabla \bullet \overline{B} = 0 \qquad (4.5.6)$$

where $\overline{D} = \varepsilon\overline{E}$ and $\overline{B} = \mu\overline{H}$. Substituting (4.5.3) into (4.5.5), and (4.5.4) into (4.5.6) yields *Laplace's equation*:

$$\nabla^2\Phi = \nabla^2\Psi = 0 \qquad \text{(Laplace's equation)} \qquad (4.5.7)$$

To find static electric or magnetic fields produced by any given set of boundary conditions we need only to solve Laplace's equation (4.5.7) for Φ or Ψ, and then use (4.5.3) or (4.5.4) to compute the gradient of the potential. One approach to solving Laplace's equation is developed in the following section.

Example 4.5A

Does the potential $\Phi = 1/r$ satisfy Laplace's equation $\nabla^2\Phi = 0$, where $r = (x^2 + y^2 + z^2)^{0.5}$?

Solution: $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$. First: $(\partial/\partial x)\ (x^2 + y^2 + z^2)^{-0.5} = -0.5(x^2 + y^2 + z^2)^{-1.5}(2x)$, so $(\partial^2/\partial x^2)\ (x^2 + y^2 + z^2)^{-0.5} = 0.75(x^2 + y^2 + z^2)^{-2.5}(2x)^2 - (x^2 + y^2 + z^2)^{-1.5}$. Therefore $(\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2)(x^2 + y^2 + z^2)^{-0.5} = 3(x^2 + y^2 + z^2)^{-2.5}(x^2 + y^2 + z^2) - 3(x^2 + y^2 + z^2)^{-1.5} = 0$. So this potential satisfies Laplace's equation. The algebra could have been simplified if instead we wrote ∇^2 in spherical coordinates (see Appendix C), because only the radial term is potentially non-zero for $\Phi = 1/r$: $\nabla^2 = r^{-2}(\partial/\partial r)(r^2\partial/\partial r)$. In this case the right-most factor is $r^2\partial r^{-1}/\partial r = r^2(-r^{-2}) = -1$, and $\partial(-1)/\partial r = 0$, so again $\nabla^2\Phi = 0$.

4.5.2 Separation of variables

We can find simple analytic solutions to Laplace's equation only in a few special cases for which the solutions can be factored into products, each of which is dependent only upon a single dimension in some coordinate system compatible with the geometry of the given boundaries. This process of separating Laplace's equation and solutions into uni-dimensional factors is called *separation of variables*. It is most easily illustrated in terms of two dimensions. Let's assume the solution can be factored:

$$\Phi(x,y) = X(x)Y(y) \tag{4.5.8}$$

Then Laplace's equation becomes:

$$\nabla^2\Phi = \partial^2\Phi/\partial x^2 + \partial^2\Phi/\partial y^2 = Y(y)\,d^2X/dx^2 + X(x)\,d^2Y/dy^2 = 0 \tag{4.5.9}$$

Dividing by X(x)Y(y) yields:

$$\left[d^2X(x)/dx^2\right]/X(x) = -\left[d^2Y(y)/dy^2\right]/Y(y) \tag{4.5.10}$$

Since (4.5.10) must be true for all values of x, y, it follows that each term must equal a constant k^2, called the *separation constant*, so that:

$$d^2X/dx^2 = -k^2X \qquad\qquad d^2Y/dy^2 = k^2Y \tag{4.5.11}$$

Generic solutions to (4.5.11) are, for $k \neq 0$:

$$X(x) = A\cos kx + B\sin kx \tag{4.5.12}$$

$$Y(y) = C\cosh ky + D\sinh ky \tag{4.5.13}$$

An equivalent alternative is $Y(y) = C'\,e^{ky} + D'\,e^{-ky}$. Generic solutions when k = 0 are:

$$X(x) = Ax + B \tag{4.5.14}$$

$$Y(y) = Cy + D \tag{4.5.15}$$

Note that by letting $k \rightarrow jk$, the sinusoidal x-dependence becomes hyperbolic, and the hyperbolic y dependence becomes sinusoidal--the roles of x and y are reversed. Whether k is zero, real, imaginary, or complex depends upon boundary conditions. Linear combinations of solutions to differential equations are also solutions to those same equations, and such combinations are often required to match boundary conditions.

These univariable solutions can be combined to yield the three solution forms for x-y coordinates:

$$\Phi(x,y) = (A + Bx)(C + Dy) \qquad \text{for } k = 0 \qquad (4.5.16)$$

$$\Phi(x,y) = (A \cos kx + B \sin kx)(C \cosh ky + D \sinh ky) \qquad \text{for } k^2 > 0 \qquad (4.5.17)$$

$$\Phi(x,y) = (A \cosh qx + B \sinh qx)(C \cos qy + D \sin qy) \qquad \text{for } k^2 < 0 \ (k = jq) \qquad (4.5.18)$$

This approach can be extended to three cartesian dimensions by letting $\Phi(x,y,z) = X(x)Y(y)Z(z)$; this leads to the solution[11]:

$$\Phi(x,y,z) = (A \cos k_x x + B \sin k_x x)(C \cos k_y y + D \sin k_y y)(E \cosh k_z z + F \sinh k_z z) \qquad (4.5.19)$$

where $k_x^2 + k_y^2 + k_z^2 = 0$. Since k_x^2, k_y^2, and k_z^2 must sum to zero, k_i^2 must be negative for one or two coordinates so that the solution is sinusoidal along either one or two axes and hyperbolic along the others.

Once the form of the solution is established, the correct form, (4.5.16) to (4.5.19), is selected and the unknown constants are determined so that the solution matches the given boundary conditions, as illustrated in the following example.

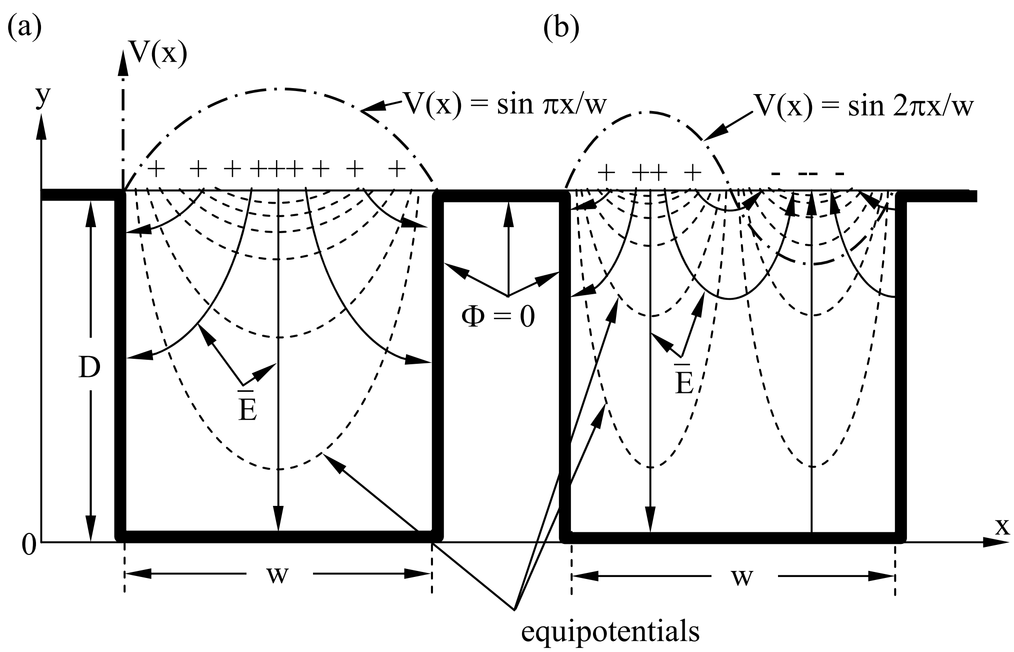

Figure 4.5.1 Static potentials and fields in a sinusoidally-driven conducting rectangular slot.

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[11] If $\Phi(x,y,z) = X(x)Y(y)Z(z)$, then $\nabla^2\Phi = YZd^2X/dx^2 + XZd^2Y/dy^2 + XYd^2Z/dz^2$. Dividing by XYZ yields $X^{-1}d^2X/dx^2 + Y^{-1}d^2Y/dy^2 + Z^{-1}d^2Z/dz^2 = 0$, which implies all three terms must be constants if the equation holds for all x,y,z; let these constants be k_x^2, k_y^2, and k_z^2, respectively. Then $d^2X(y)/dx^2 = k_x^2X(x)$, and the solution (4.5.19) follows when only $k_z^2 > 0$.

Consider an infinitely long slot of width w and depth d cut into a perfectly conducting slab, and suppose the cover to the slot has the voltage distribution V(x) = 5 sin(πx/w) volts, as illustrated in Figure 4.5.1(a). This is a two-dimensional cartesian-coordinate problem, so the solution (4.5.17) is appropriate, where we must ensure this expression yields potentials that have the given voltage across the top of the slot and zero potential over the side and bottom boundaries of the slot. Thus:

$$\Phi(x,y) = A\sin(\pi x/w)\sinh(\pi y/w) \quad [\text{volts}] \tag{4.5.20}$$

where the sine and sinh options[12] from (4.5.17) were chosen to match the given potentials on all four boundaries, and where $A = 5/\sinh(\pi D/w)$ in order to match the given potential across the top of the slot.

Figure 4.5.1(b) illustrates the solution for the case where the potential across the open top of the slot is given as V(x) = sin 2πx/w. If an arbitrary voltage V(x) is applied across the opening at the top of the slot, then a sum of sine waves can be used to match the boundary conditions.

Although all of these examples were in terms of static electric fields $\overline{E}$ and potentials Φ, they equally well could have been posed in terms of static $\overline{H}$ and magnetic potential Ψ; the forms of solutions for Ψ are identical.

Example 4.5B

A certain square region obeys $\nabla^2\Phi = 0$ and has Φ = 0 along its two walls at x = 0 and at y = 0. Φ = V volts at the isolated corner x = y = L. Φ increases linearly from 0 to V along the other two walls. What are Φ(x,y) and $\overline{E}(x,y)$ within the square?

Solution: Separation of variables permits linear gradients in potentials in rectangular coordinates via (4.5.14) and (4.5.15), so the potential can have the form Φ = (Ax + B)(Cy + D) where B = D = 0 for this example. Boundary conditions are matched for $\Phi(x,y) = (V/L^2)xy$ [V]. It follows that: $\overline{E} = -\nabla\Phi = (V/L^2)(\hat{x}y + \hat{y}x)$.

4.5.3 Separation of variables in cylindrical and spherical coordinates

Laplace's equation can be separated only in four known coordinate systems: cartesian, cylindrical, spherical, and elliptical. Section 4.5.2 explored separation in cartesian coordinates, together with an example of how boundary conditions could then be applied to determine a total solution for the potential and therefore for the fields. The same procedure can be used in a few other coordinate systems, as illustrated below for cylindrical and spherical coordinates.

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[12] $\sinh x = (e^{x} - e^{-x})/2$ and $\cosh x = (e^{x} + e^{-x})/2$.

When there is no dependence on the z coordinate, Laplace's equation in cylindrical coordinates reduces to circular coordinates and is:

$$\nabla^2\Phi = r^{-1}(\partial/\partial r)(r\partial\Phi/\partial r) + r^{-2}\left(\partial^2\Phi/\partial\phi^2\right) = 0 \qquad (4.5.21)$$

Appendix C reviews the del operator in several coordinate systems. We again assume the solution can be separated:

$$\Phi = R(r)\Phi(\phi) \qquad (4.5.22)$$

Substitution of (4.5.22) into (4.5.21) and dividing by $R(r)\Phi(\phi)$ yields:

$$R^{-1}(d/dr)(r\ dR/dr) = -\Phi^{-1}\left(d^2\Phi/d\phi^2\right) = m^2 \qquad (4.5.23)$$

where m^2 is the separation constant.

The solution to (4.5.23) depends on whether m^2 is zero, positive, or negative:

$$\Phi(r,\phi) = [A + B\phi][C + D\ (\ln r)] \qquad (\text{for } m^2 = 0) \qquad (4.5.24)$$

$$\Phi(r,\phi) = (A \sin m\phi + B \cos m\phi)(Cr^m + Dr^{-m}) \qquad (\text{for } m^2 > 0) \qquad (4.5.25)$$

$$\Phi(r,\phi) = [A \sinh p\phi + B \cosh p\phi][C \cos(p \ln r) + D \sin(p \ln r)] \qquad (\text{for } m^2 < 0) \qquad (4.5.26)$$

where A, B, C, and D are constants to be determined and $m \equiv jp$ for $m^2 < 0$.

A few examples of boundary conditions and the resulting solutions follow. The simplest case is a uniform field in the $+\hat{x}$ direction; the solution that matches these boundary conditions is (4.5.25) for m = 1:

$$\Phi(r,\phi) = Br \cos\phi \qquad (4.5.27)$$

Another simple example is that of a conducting cylinder of radius R and potential V. Then the potential inside the cylinder is V and that outside decays as ln r, as given by (1.3.12), when m = C = 0:

$$\Phi(r,\phi) = (V/\ln R)\ln r \qquad (4.5.28)$$

The electric field associated with this electric potential is:

$$\overline{E} = -\nabla\Phi = -\hat{r}\partial\Phi/\partial r = \hat{r}(V/\ln R)r^{-1} \qquad (4.5.29)$$

Thus $\overline{E}$ is radially directed away from the conducting cylinder if V is positive, and decays as r^{-1}.

A final interesting example is that of a dielectric cylinder perpendicular to an applied electric field $\overline{E} = \hat{x}E_o$. Outside the cylinder the potential follows from (4.5.25) for m = 1 and is:

$$\Phi(r,\phi) = -E_o r\cos\phi + (AR/r)\cos\phi \tag{4.5.30}$$

The potential inside can have no singularity at the origin and is:

$$\Phi(r,\phi) = -E_o(Br/R)\cos\phi \tag{4.5.31}$$

which corresponds to a uniform electric field. The unknown constants A and B can be found by matching the boundary conditions at the surface of the dielectric cylinder, where both Φ and $\overline{D}$ must be continuous across the boundary between regions 1 and 2. The two linear equations for continuity ($\Phi_1 = \Phi_2$, and $\overline{D}_1 = \overline{D}_2$) can be solved for the two unknowns A and B. The electric fields for this case are sketched in Figure 4.5.2.

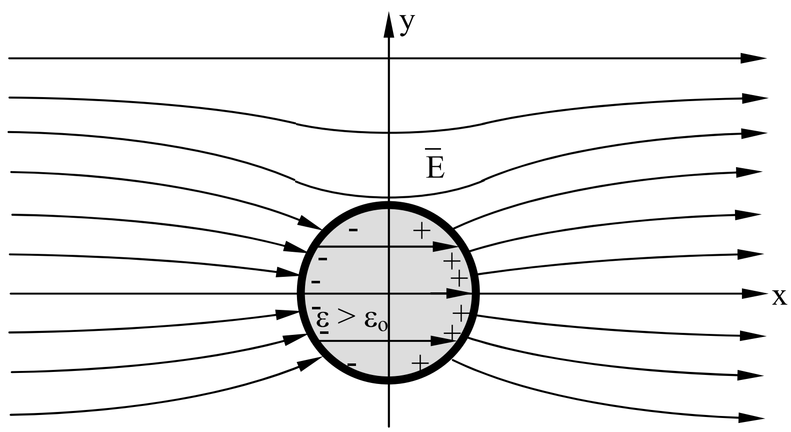

Figure 4.5.2 Electric fields perpendicular to a dielectric cylinder.

If these cylindrical boundary conditions also vary with z, the solution to Laplace's equation becomes:

$$\Phi(r,\phi,z) = \Phi_o[C_1e^{kz} + C_2e^{-kz}][C_3 \cos n\phi + C_4 \sin n\phi][C_5J_n(kr) + C_6N_n(kr)] \tag{4.5.32}$$

where J_n and N_n are Bessel functions of order n of the first and second kind, respectively, and C_i are dimensionless constants that match the boundary conditions. The rapidly growing complexity of these solutions as the dimensionality of the problem increases generally mandates numerical solutions of such boundary value problems in practical cases.

Our final example involves spherical coordinates, for which the solutions are:

$$\Phi(r,\theta,\phi) = \Phi_o[C_1r^n + C_2r^{-n-1}][C_3 \cos m\phi + C_4 \sin m\phi][C_5P_n^m(\cos\theta) + C_6Q_n^m(\cos\theta)] \tag{4.5.33}$$

where P_n^m and Q_n^m are associated Legendre functions of the first and second kind, respectively, and C_i are again dimensionless constants chosen to match boundary conditions. Certain spherical problems do not invoke Legendre functions, however, as illustrated below.

A dielectric sphere inserted in a uniform electric field $\hat{x}E_o$ exhibits the same general form of solution as does the dielectric rod perpendicular to a uniform applied electric field; the solution is the sum of the applied field and the dipole field produced by the induced polarization charges on the surface of the rod or sphere. Inside the sphere the field is uniform, as suggested in Figure 4.5.2. Polarization charges are discussed more fully in Section 2.5.3. The potential follows from (4.5.33) with n = 1 and m = 0, and is simply:

$$\Phi(r,\theta,\phi) = -E_o \cos\theta\left(C_1 r - C_2 R^3 r^{-2}\right) \tag{4.5.34}$$

where $C_2 = 0$ inside, and for the region outside the cylinder C_2 is proportional to the induced electric dipole. C_1 outside is unity and inside diminishes below unity as ε increases.

If the sphere in the uniform electric field is conducting, then in (4.5.34) $C_1 = C_2 = 0$ inside the sphere, and the field there is zero; the surface charge is:

$$\rho_s = -\varepsilon_o \hat{n} \bullet \nabla\Phi\big|_{r=R} = \varepsilon_o E_r = 3\varepsilon_o E_o \cos\theta \ \left[\mathrm{Cm^{-2}}\right] \tag{4.5.35}$$

Outside the conducting sphere $C_1 = 1$, and to ensure $\Phi(r = R) = 0$, C_2 must also be unity.

The same considerations also apply to magnetic potentials. For example, a sphere of permeability μ and radius R placed in a uniform magnetic field would also have an induced magnetic dipole that produces a uniform magnetic field inside, and produces outside the superposition of the original uniform field with a magnetic dipole field produced by the sphere. A closely related example involves a sphere of radius R having surface current:

$$\overline{J}_s = \hat{\phi}\sin\theta \ \left[\mathrm{Am^{-1}}\right] \tag{4.5.36}$$

This can be produced approximately by a coil wound on the surface of the sphere with a constant number of turns per unit length along the z axis.

For a permeable sphere in a uniform magnetic field $\overline{H} = -\hat{z}H_o$, the solution to Laplace's equation for magnetic potential $\nabla^2\Psi = 0$ has a form similar to (4.5.34):

$$\Psi(r,\theta) = Cr\cos\theta \qquad \text{(inside the sphere; } r < R) \tag{4.5.37}$$

$$\Psi(r,\theta) = Cr^{-2}\cos\theta + H_o r\cos\theta \qquad \text{(outside the sphere; } r > R) \tag{4.5.38}$$

Using $\overline{H} = -\nabla\Psi$, we obtain:

$$\overline{H}(r,\theta) = -\hat{z}C \qquad \text{(inside the sphere; r < R)} \tag{4.5.39}$$

$$\overline{H}(r,\theta) = -C(R/r)^2(\hat{r}\cos\theta + 0.5\hat{\theta}\sin\theta) - \hat{z}H_o \qquad \text{(outside the sphere; r > R)} \tag{4.5.40}$$

Matching boundary conditions at the surface of the sphere yields C; e.g. equate $\overline{B} = \mu\overline{H}$ inside to $\overline{B} = \mu_o H$ outside by equating (4.5.39) to (4.5.40) for $\theta = 0$.

4.6 Flux tubes and field mapping

4.6.1 Static field flux tubes

Flux tubes are arbitrarily designated bundles of static electric or magnetic field lines in charge-free regions, as illustrated in Figure 4.6.1.

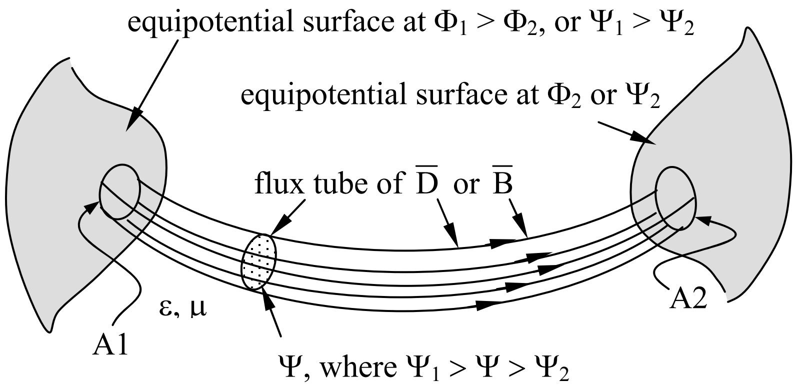

Figure 4.6.1 Electric or magnetic flux tube between two equipotential surfaces.

The divergence of such static fields is zero by virtue of Gauss's laws, and their curl is zero by virtue of Faraday's and Ampere's laws. The integral forms of Gauss's laws, (2.4.17) and (2.4.18), say that the total electric displacement $\overline{D}$ or magnetic flux $\overline{B}$ crossing the surface A of a volume V must be zero in a charge-free region:

$$\oiint_A (\overline{D} \bullet \hat{n})\, da = 0 \tag{4.6.1}$$

$$\oiint_A (\overline{B} \bullet \hat{n})\, da = 0 \tag{4.6.2}$$

Therefore if the walls of flux tubes are parallel to the fields then the walls contribute nothing to the integrals (4.6.1) and (4.6.2) and the total flux entering the area A1 of the flux tube at one end (A1) must equal that exiting through the area A2 at the other end, as illustrated:

$$\oiint_{A1} (\overline{D} \bullet \hat{n})\, da = -\oiint_{A2} (\overline{D} \bullet \hat{n})\, da \tag{4.6.3}$$

$$\oiint_{A1}(\overline{B}\bullet\hat{n})\,da = -\oiint_{A2}(\overline{B}\bullet\hat{n})\,da \tag{4.6.4}$$

Consider two surfaces with potential differences between them, as illustrated in Figure 4.6.1. A representative flux tube is shown and all other fields are omitted from the figure. The field lines could correspond to either $\overline{D}$ or $\overline{B}$. Constant ε and μ are not required for $\overline{D}$ and $\overline{B}$ flux tubes because $\overline{D}$ and $\overline{B}$ already incorporate the effects of inhomogeneous media. If the permittivity ε and permeability μ were constant then the figure could also apply to $\overline{E}$ or $\overline{H}$, respectively.

4.6.2 Field mapping

$\overline{E}$ and $\overline{H}$ are gradients of the potentials Φ and Ψ, respectively [see (4.6.2) and (4.6.5)], and therefore the equipotential surfaces are perpendicular to their corresponding fields, as suggested in Figure 4.6.1. This orthogonality leads to a useful technique called *field mapping* for sketching approximately correct field distributions given arbitrarily shaped surfaces at known potentials. The method is particularly simple for "two-dimensional" geometries that depend only on the x,y coordinates and are independent of z, such as the pair of circular surfaces illustrated in Figure 4.6.2(a) and the pair of ovals in Figure 4.6.2(b). Assume that the potential of the inner surface is Φ_1 or Ψ_1, and that at the outer surface is Φ_2 or Ψ_2.

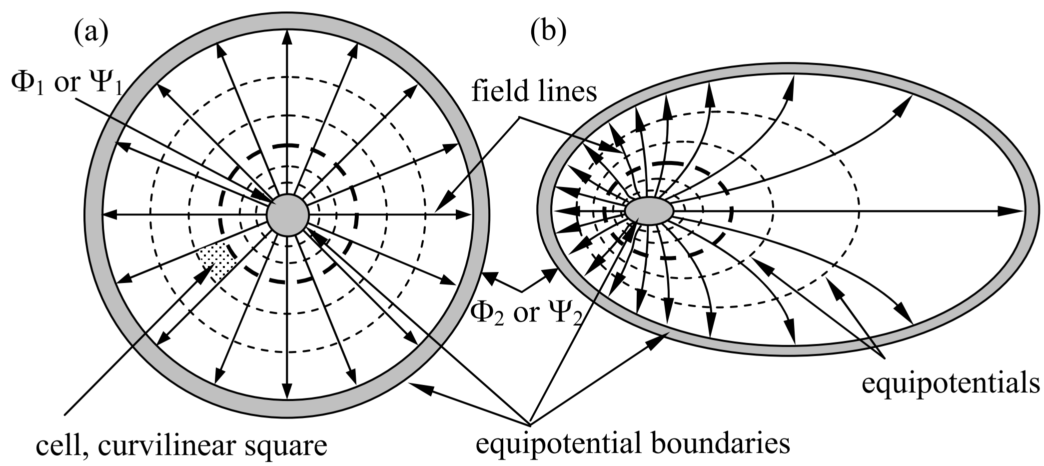

Figure 4.6.2 Field mapping of static electric and magnetic fields.

Because: 1) the lateral spacing between adjacent equipotential surfaces and (in two-dimensional geometries) between adjacent field lines are both inversely proportional to the local field strength, and 2) the equipotentials and field lines are mutually orthogonal, it follows that the rectangular shape of the cells formed by these adjacent lines is preserved over the field even as the field strengths and cell sizes vary. That is, the curvilinear square illustrated in Figure 4.6.2(a) has approximately the same shape (but not size) as all other cells in the figure, and approaches a perfect square as the cells are subdivided indefinitely. If sketched perfectly, any two-dimensional static potential distribution can be subdivided indefinitely into such curvilinear square cells.

One algorithm for performing such a subdivision is to begin by sketching a first-guess equipotential surface that: 1) separates the two (or more) equipotential boundaries and 2) is orthogonal to the first-guess field lines, which also can be sketched. These field lines must be othogonal to the equipotential boundaries. For example, this first sketched surface might have potential $(\Phi_1 + \Phi_2)/2$, where Φ_1 and Φ_2 are the applied potentials. The spacing between the initially sketched field lines and between the initial equipotential surfaces should form approximate curvilinear squares. Each such square can then be subdivided into four smaller curvilinear squares using the same algorithm. If the initial guesses were correct, then the curvilinear squares approach true squares when infinitely subdivided. If they do not, the first guess is revised appropriately and the process can be repeated until the desired insight or perfection is achieved. In general there will be some fractional squares arranged along one of the field lines, but these become negligible in the limit.

Figure 4.6.2(a) illustrates how the flux tubes in a co-axial geometry are radial with field strength inversely proportional to radius. Therefore, when designing systems limited by the maximum allowable field strength, one avoids incorporating surfaces with small radii of curvature or sharp points. Figure 4.6.2(b) illustrates how the method can be adapted to arbitrarily shaped boundaries, albeit with more difficulty. Computer-based algorithms using relaxation techniques can implement such strategies rapidly for both two-dimensional and three-dimensional geometries. In three dimensions, however, the spacing between field lines varies inversely with the square root of their strength, and so the height-to-width ratio of the curvilinear 3-dimensional rectangles formed by the field lines and potentials is not preserved across the structure.