ACOUSTIC WAVES IN GASES

Basic Differences with EM Waves:

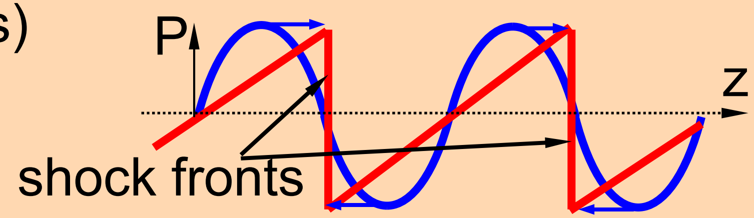
Electromagnetic Waves	Acoustic Waves
$\overline{E}$, $\overline{H}$ are vectors $\perp$ $\overline{S}$	$\overline{U}$(velocity) // $\overline{S}$, P(pressure) is scalar
Linear physics	Non-linear physics, use perturbations

Acoustic Non-linearities:

Compression heats the gas; cooling by conduction and radiation (adiabatic assumption—no heat transfer)

Compression and advection introduce position shifts in wave

Wave velocity depends on pressure, varies along wave (loud sounds form shock waves)



Acoustic Variables:

Pressure: $P\left[Nm^{-2}\right] = P_o + p$

Velocity: $\overline{U}\left[ms^{-1}\right] = \overline{U}_o + \overline{u} = \overline{u}\ \left(\text{set } \overline{U}_o = 0 \text{ here}\right)$

Density: $\rho\left[kg\ m^3\right] = \rho_o + \rho_1$

use perturbations

ACOUSTIC EQUATIONS

Mass Conservation Equation:

Recall: $\nabla \bullet \bar{J} = \nabla \bullet \rho_e \bar{u} = -\frac{\partial \rho_e}{\partial t}$ Conservation of charge

Acoustics: $\nabla \bullet \rho \bar{u} = -\frac{\partial \rho}{\partial t}$ Conservation of mass

Linearize: $\nabla \bullet (\rho_o + \rho_1)(\bar{U}_o + \bar{u}) \cong -\frac{\partial(\rho_o + \rho_1)}{\partial t} = -\frac{\partial \rho_1}{\partial t}$ ($\bar{U}_o \to 0$)

Drop 2nd order term $(\rho_1 \bar{u})$

Linearized Conservation of Mass:

$$\rho_o \nabla \bullet \bar{u} \cong -\frac{\partial \rho_1}{\partial t}$$

Linearized Force Equation (f = ma):

$$\nabla p = -\rho_o \frac{\partial \bar{u}}{\partial t}$$

Constitutive Equation:

Fractional changes in gas density and pressure are proportional:

$$\frac{d\rho}{\rho} = \frac{1}{\gamma}\frac{dP}{P} \quad \Rightarrow \quad \rho_1 = \left(\frac{\rho_o}{\gamma P_o}\right)p$$

"adiabatic exponent" γ = 5/3 monotomic gas, ~1.4 air, 1-2 else

3 Equations, 3 Unknowns:

Reduce to 2 unknowns $(p, \bar{u})$

ACOUSTIC EQUATIONS

Differential Equations:

Newton's Law (f = ma): $\nabla p = -\rho_o \frac{\partial \bar{u}}{\partial t}$ [Nm^{-3}] [$kg\ m^{-2} s^{-2}$]

Conservation of Mass: $\nabla \bullet \bar{u} \cong -\frac{1}{\gamma P_o} \frac{\partial p}{\partial t}$ [s^{-1}]

Acoustic Wave Equation:

$$\nabla \bullet \nabla p \Rightarrow \underbrace{\nabla^2 p}_{} - \underbrace{\frac{\rho_o}{\gamma P_o} \frac{\partial^2 p}{\partial t^2}}_{} = 0$$ "Acoustic Wave Equation"

2nd spatial derivative = 2nd derivative in time

Solution:

$p(t, \bar{r}) = p(\omega t - \bar{k} \bullet \bar{r})$ [Nm^{-2}]

UNIFORM PLANE WAVES

Example: assume p(t,r) = cos(ωt-kz):

$$\nabla p = -\rho_o \frac{\partial \bar{u}}{\partial t} \Rightarrow \bar{u} = -\hat{z}\int \frac{1}{\rho_o}\nabla p \, dt = \hat{z}\underbrace{\frac{k}{\rho_o \omega}}_{\eta_s^{-1}}\cos(\omega t - kz)$$

Substituting solution into wave equation

⇒ "Acoustic Dispersion Relation": $k = \omega\sqrt{\frac{\rho_o}{\gamma P_o}} = \frac{\omega}{c_s}$

Acoustic Impedance of Gas:

$\eta_s = \frac{\omega \rho_o}{k} = \sqrt{\rho_o \gamma P_o}$ ($\eta_s \cong 425$ Nsm^{-3} [≠Ω] for air 20°C)

Velocity of Sound:

Phase velocity: $v_p = \frac{\omega}{k} = \sqrt{\frac{\gamma P_o}{\rho_o}} = c_s$

Group velocity: $v_g = \left(\frac{\partial k}{\partial \omega}\right)^{-1} = \sqrt{\frac{\gamma P_o}{\rho_o}} = c_s$

Example:

Air at 0°C, Surface at P_o

⇒ $c_s \cong 330$ m/s

(G = 1.4, $\rho_o = 1.29$ kg/m^3

$P_o = 1.01 \times 10^5$ N/m^2)

Velocity of Sound in Liquids and Solids:

$c_s = (K/\rho_o)^{0.5} \cong 1{,}500$ ms^{-1} in water, ≅ 1,500 – 13,000 in solids

"Bulk modulus"

ACOUSTIC POWER AND ENERGY

Poynting Theorem, differential form:

Recall: $\nabla p = -\rho_o \frac{\partial \bar{u}}{\partial t}$ [Nm^{-3}] [kg m^{-2}s^{-2}] $\qquad \nabla \bullet \bar{u} \cong -\frac{1}{\gamma P_o}\frac{\partial p}{\partial t}$ [s^{-1}]

Note: Wave intensity [Wm^{-2}] $= p\bar{u}$ [(Nm^{-2})(ms^{-1})]
[Watts]

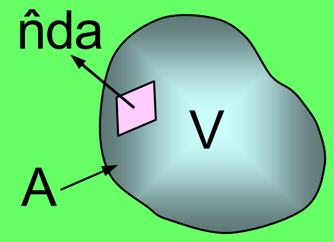
Try: $\nabla \bullet \bar{u}p = \bar{u} \bullet \nabla p + p \nabla \bullet \bar{u} = -\rho_o \bar{u} \bullet \frac{\partial \bar{u}}{\partial t} - \frac{1}{\gamma P_o} p \frac{\partial p}{\partial t}$

$$\nabla \bullet \bar{u}p = -\frac{1}{2}\frac{\partial}{\partial t}\left[\rho_o |\bar{u}|^2 - \frac{1}{\gamma P_o} p^2\right] \quad [\text{W/m}^3]$$ Acoustic Poynting Theorem

Integral form:

$$\int_A p\bar{u} \bullet \hat{n}\,da = -\frac{\partial}{\partial t}\int_V \left[\rho_o \frac{|\bar{u}|^2}{2} + \frac{p^2}{2\gamma P_o}\right] dv$$

I[Wm^{-2}] — Acoustic intensity; W_k [Jm^{-3}] — Kinetic; W_p — Potential



Plane Wave Intensity $I = p\bar{u} \bullet \hat{n}$ [Wm^{-2}]:

Intensity: $I(t) = \frac{p^2}{\eta_s} = \eta_s |\bar{u}|^2$ [Wm^{-2}], $\qquad I_o = \langle I(t) \rangle = \frac{|\underline{p}_o|^2}{2\eta_s} = \eta_s \frac{|\underline{\bar{u}}_o|^2}{2}$

Where: $\eta_s = \sqrt{\rho_o \gamma P_o}$ {$\cong$ 425 Nsm^{-3} in surface air}

ACOUSTIC INTENSITY

Plane Wave Intensity $I = p\bar{u} \bullet \hat{n} \left[Wm^{-2} \right]$:

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Where: $\eta_s = \sqrt{\rho_o \gamma P_o}$ {≅ 425 Nsm^{-3} in surface air}

Example: small radio at beach:

$I_o = 1\ [Wm^{-2}]$ at 1 kHz

$\Rightarrow p_o = \sqrt{2\eta_s I_o} = \sqrt{850} = \sim 30\ [N/m^2]$

$u_o = p_o/\eta_s = 0.07\ [ms^{-1}]$; $\Delta z = 2u_o/\omega = 10$ microns

Example: Threshold of hearing:

$I_{thresh} \cong 0$ dB (acoustic scale) $= 10^{-12}\ [Wm^{-2}]$

$p_o = \sqrt{2\eta_s I_o} = \sqrt{850 \times 10^{-12}} = \sim 3 \times 10^{-5}\ [N/m^2]$

$u_o = p_o/\eta_s = 3\times10^{-5}/425 \cong 7 \times 10^{-8}\ [ms^{-1}]$

$\Delta z \cong 2\frac{u_o}{\omega} \cong 2\frac{7\times10^{-8}}{7\times10^{3}} = 2\times10^{-11}$ [m] = 0.2 Å (< atom)

BOUNDARY CONDITIONS

Interfaces between gases or liquids:

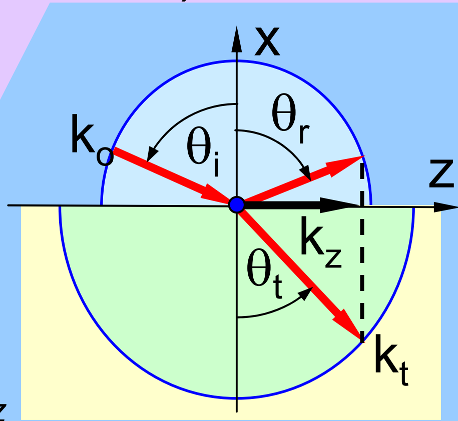
Pressure: $\Delta p = 0$ (otherwise ∞ acceleration of zero-mass boundary)

Velocity: $\Delta u_\perp = 0$ (otherwise ∞ mass-density accumulation)

Rigid Boundaries:

Pressure: Unconstrained

Velocity: $\Delta u_\perp = 0$ (rigid body is motionless)



Reflection at Non-Rigid Boundaries:

Incident wave: $\underline{p}_i = p_o e^{-j\bar{k}_i \cdot \bar{r}} = p_o e^{+jk_o \cos\theta_i x - jk_o \sin\theta_i z}$

Reflected wave: $\underline{p}_r = \underline{p}_r e^{-j\bar{k}_r \cdot \bar{r}} = \underline{p}_{ro} e^{-jk_o \cos\theta_r x - jk_o \sin\theta_r z}$

Transmitted wave: $\underline{p}_t = \underline{p}_t e^{-j\bar{k}_t \cdot \bar{r}} = \underline{p}_{to} e^{+jk_t \cos\theta_t x - jk_t \sin\theta_t z}$

Velocities: Same, but $\underline{p}_{o,ro,to} \rightarrow \underline{\bar{u}}_{o,ro,to}$, $\underline{p}_{r,t} \rightarrow \underline{\bar{u}}_{r,t}$

Matching Phases: $\Rightarrow$

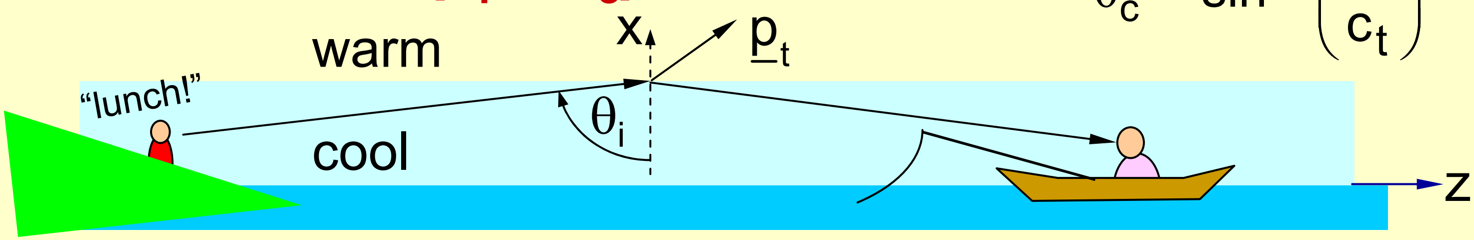
Snell's Law: $\sin\theta_i / \sin\theta_t = k_t / k_i = \sqrt{\dfrac{\rho_t \gamma_i}{\rho_i \gamma_t}}$

$k = \omega \sqrt{\dfrac{\rho_o}{\gamma P_o}}$ $\theta_r = \theta_i$ Critical angle: $\theta_c = \sin^{-1}\left(\dfrac{c_i}{c_t}\right) = \sin^{-1}\sqrt{\dfrac{\rho_t \gamma_i}{\rho_i \gamma_t}}$

REFLECTIONS AT BOUNDARIES

Evanescent Waves ($\theta_i > \theta_c$):

$$\theta_c = \sin^{-1}\left(\frac{c_i}{c_t}\right)$$

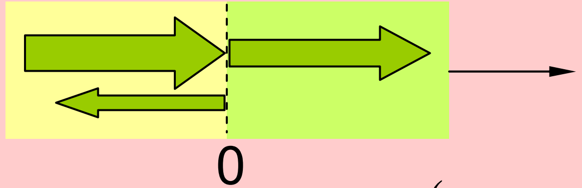


Recall: $\underline{p}_t = \underline{p}_{to} e^{-j(k_t \cos\theta_t)x - j(k_t \sin\theta_t)z} = \underline{p}_{to} e^{-\alpha x - j(k_t \sin\theta_t)z}$

Where: $\cos\theta_t = \sqrt{1 - \sin^2\theta_t}$ and $\sin\theta_t = \sqrt{\rho_i\gamma_t / \rho_t\gamma_i}\ \sin\theta_i > 1$

So: $k_t \cos\theta_t = \pm j\alpha$

Normal Incidence, two gases:



$\underline{p}$: $\underline{p}_i e^{-jk_o z} + \underline{p}_i \underline{\Gamma} e^{+jk_o z} = \underline{p}_i \underline{T} e^{-jk_t z} \rightarrow 1 + \underline{\Gamma} = \underline{T}$ at $z = 0$ $(\Delta p = 0)$

$\overline{u}$: $\underline{p}_i/\eta_o - \underline{p}_i \underline{\Gamma}/\eta_o = \underline{p}_i \underline{T}/\eta_t \rightarrow 1 - \underline{\Gamma} = \underline{T}\eta_o/\eta_t$ at $z = 0$ $(\Delta \overline{u}_\perp = 0)$

Solving: $\underline{T} = 2\eta_t/(\eta_t + \eta_o)$ where $\eta_o = \omega\rho_o/k = \sqrt{\rho_o \gamma P_o}$

Reflections from Solid Surface $(\hat{n} \bullet \overline{u} = 0)$: