6.013/ESD.013J Electromagnetics and Applications, Fall 2005

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6.013/ESD.013J — Electromagnetics and Applications Fall 2005

Problem Set 7 - Solutions

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Problem 7.1

A

$$i(z,t) = \frac{C}{\Delta z}\frac{d}{dt}[v(z-\Delta z) - v(z,t)]; \quad v(z,t) = \frac{L}{\Delta z}\frac{d}{dt}[i(z) - i(z+\Delta z)]$$

$$\lim_{\Delta z \to 0} i(z,t) = -C\frac{\partial^2 v}{\partial t \partial z}; \quad v(z,t) = -L\frac{\partial^2 i}{\partial t \partial z}$$

B

$$i(z,t) = \text{Re}\left[\hat{\imath} e^{j(\omega t - kz)}\right], \quad v(z,t) = \text{Re}\left[\hat{v} e^{j(\omega t - kz)}\right]$$

$$\hat{\imath} = -C\omega k \hat{v}; \quad \hat{v} = -L\omega k \hat{\imath}$$

$$\hat{\imath} = +LC\omega^2 k^2 \hat{\imath} \implies LC\omega^2 k^2 = 1 \implies k = \frac{1}{\omega\sqrt{LC}}$$

C

$$v_p = \frac{\omega}{k} = \omega^2\sqrt{LC}$$

$$v_g = \frac{d\omega}{dk} = -\omega^2\sqrt{LC}$$

Such systems are called backward wave because the group velocity is opposite in direction to the phase velocity.

D

$$\hat{v}(z) = V_1 \sin kz + V_2 \cos kz$$

$$\hat{v}(z=0) = 0 = V_2$$

$$\hat{v}(z=-l) = V_0 = -V_1 \sin kl \implies \hat{v}(z) = \frac{-V_0}{\sin kl}\sin kz$$

$$\hat{\imath}(z) = -Cj\omega\frac{d\hat{v}}{dz} = \frac{j\omega C V_0 k \cos kz}{\sin kl} = j\sqrt{\frac{C}{L}}V_0\frac{\cos kz}{\sin kl}$$

E

$$\text{Resonance} \implies \sin kl = 0 \implies kl = n\pi \implies \omega_n = \frac{1}{\left(\frac{n\pi}{l}\right)\sqrt{LC}}$$

Problem 7.2

A

$$Z_{L,n} = Z_n(z=0) = 0.5(1+j) \implies Z_n(z=-\lambda/4) = \frac{1}{Z_n(z=0)} = \frac{1}{0.5(1+j)}$$

$$Y_n(z=-\lambda/4) = 0.5(1+j) \implies Y(z=-\lambda/4) = 0.01(1+j)$$

$$Y_T = (jB) \; // \; Y(z=-\lambda/4) = jB + 0.01(1+j) = 0.01 + j(0.01+B)$$

$$\implies B = -0.01 \quad \text{(inductive susceptances are negative)}$$

$$R_T = \frac{1}{G_T} = \frac{1}{0.01} = 100$$

The resistance is the reciprocal of the conductance. To maximize the power delivered to the load,

$$R_S = R_T = 100$$

B

For a short circuit

$$Z_n(z=-l) = j\tan(kl) \implies Y_n = -j\cot(kl)$$

$$\cot(2\pi l/\lambda) = 0.01 \implies l = \frac{\lambda}{2\pi}\cot^{-1}(0.01)$$

$$a = \frac{1}{2\pi}\cot^{-1}(0.01) = 0.248 = l/\lambda$$

C

$$\langle P_L \rangle = \frac{1}{2}\frac{(V_0/2)^2}{R_S} = \frac{1}{8}\frac{V_0^2}{R_S}$$

Hence, the power dissipated in a matched load is $V_0^2/800$.

D

We have that $c = \lambda f$, but the speed of light is constant, so doubling the frequency gives

$$\lambda_{\text{new}} = \frac{\lambda_{\text{old}}}{2}.$$

Since λ is smaller and the line is half a wavelength long, the length of the transmission line (in meters) is also constant

$$l = \frac{\lambda_{\text{old}}}{4} = \frac{\lambda_{\text{new}}}{2}.$$

$$Z_n(z=-\lambda/2) = Z_n(z=0) = 0.5(1+j)$$

$$Y_n(z=-\lambda/2) = \frac{2}{1+j} = 1-j \implies Y(z=-\lambda/2) = 0.02(1-j)$$

$$Y_T = jB \; // \; Y(z=-\lambda/2) = 0.02 + j(B-0.02)$$

$$B = 0.02 \quad \text{(capacitive susceptance is positive)}$$

For a matched circuit, to maximize the power delivered to the load

$$R_S = R_T = \frac{1}{G_T} = 50.$$

To find the length of the line required to make this capacitance out of a short circuited line

$$-\cot(2\pi l/\lambda) = 0.02 \implies l = \frac{\lambda}{2\pi}\cot^{-1}(-0.02)$$

Hence, $l = 0.253\lambda \implies l = 0.253\lambda \implies a = 0.253$. Finally, the power into a matched load is

$$\langle P_L \rangle = \frac{1}{8}\frac{V_0^2}{R_S} = \frac{V_0^2}{400}.$$

Problem 7.3

A

With the switch open, looking into the first $\lambda/4$ transformer,

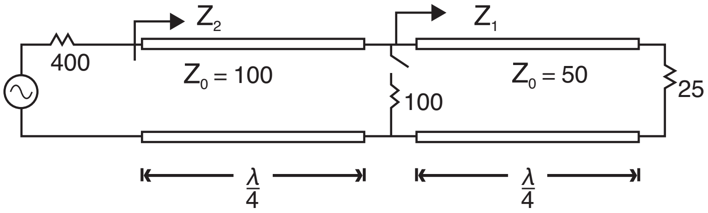

Figure 1: $\lambda/4$ Transformers. (Image by MIT OpenCourseWare.)

$$Z_1 = \left(\frac{25}{50}\right)^{-1}(50) = 100.$$

The second $\lambda/4$ transformer has 100Ω hanging on the end, and is therefore matched giving $100\Omega = Z_2$ as the equivalent load impedance at the source.

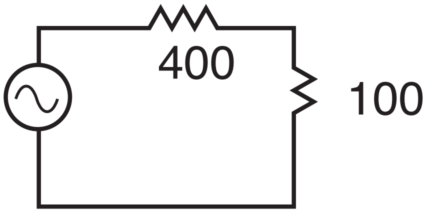

Figure 2: Equivalent circuit at source end of transmission line system. (Image by MIT OpenCourseWare.)

$$\langle P_{\text{Source}} \rangle = \frac{1}{2}\left(\frac{V_0}{500}\right)(V_0) = \frac{V_0^2}{1000}$$

With the switch closed, Z_1 is still 100; however, now the load on the second $\lambda/4$ transformer is 100 // 100 = 50Ω

$$Z_2 = \left(\frac{50}{100}\right)^{-1}(100) = 200$$

$$\langle P_{\text{Source}}\rangle = \frac{1}{2}\left(\frac{V_0}{600}\right)(V_0) = \frac{V_0^2}{1200}$$

B

In part A, we computed the total time-averaged power from the source. With the switch open, all the power that enters the transmission line must be dissipated in R_L.

$$\langle P_L\rangle = \frac{1}{5}\langle P_{\text{Source}}\rangle = \frac{V_0^2}{5000}$$

With the switch closed, the load on the first $\lambda/4$ transformer looks like two 100 ohm resistors in parallel

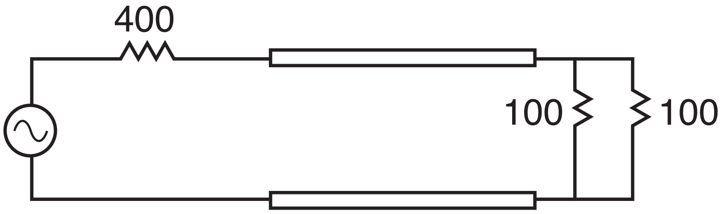

Figure 3: $\lambda/4$ Transformer. (Image by MIT OpenCourseWare.)

$\Longrightarrow$ half the power goes into the resistor on the switch and the other half into the second $\lambda/4$ transformer. All the power that goes into the second $\lambda/4$ transformer goes into R_L.

$$\langle P_L\rangle = \frac{1}{2}\frac{1}{3}\langle P_{\text{Source}}\rangle = \frac{V_0^2}{7200}$$

Problem 7.4

A

The fraction of the power reflected is $A = |\Gamma_L|^2$.

B

$$Z_L = 100 + 100j$$

$$\Longrightarrow Z_{LN} = \frac{Z_L}{Z_0} = 1 + j$$

$$\Gamma_L = \frac{Z_{LN} - 1}{Z_{LN} + 1} = \frac{j}{2+j} = \frac{1+2j}{5} = 0.2 + 0.4j$$

C

We need to rotate Γ by $(90-\beta)$ degrees, so $2kl = (90-\beta)\frac{\pi}{180}$ for Γ to line up with the real axis.

$$l = \frac{(90-\beta)}{4}\frac{\lambda}{180} = \frac{(90-\beta)}{720}\lambda = q\lambda, \quad q = \left(\frac{90-\beta}{720}\right)$$

D

The impedance at the load side of the quarter wave transformer is KZ_0. The impedance looking into the terminals of the quarter wave transformer on the generator's side is $(KZ_0/Z_T)^{-1}Z_T = Z_T^2/KZ_0$ to match the transmission line on the generator side.

$$\frac{Z_T^2}{KZ_0} = Z_0 \implies Z_T = Z_0\sqrt{K},\ Z_0 = 100\Omega$$

E

$$|\Gamma| = |\Gamma_L| = \frac{1}{\sqrt{5}} = \Gamma \implies K = \frac{1+|\Gamma_L|}{1-|\Gamma_L|} = \frac{1+1/\sqrt{5}}{1-1/\sqrt{5}}$$

$$K = R_n = \frac{1+\Gamma}{1-\Gamma} = \frac{1+1/\sqrt{5}}{1-1/\sqrt{5}}$$

F

If we continue to rotate Γ until it aligns with the negative real axis, i.e. by $(270-\beta)$ degrees

$$\Gamma = -|\Gamma_L| = -\frac{1}{\sqrt{5}} \implies K = R_n = \frac{1-1/\sqrt{5}}{1+1/\sqrt{5}}$$

$$Z_T = Z_0\sqrt{K}$$

$$q = \left(\frac{270-\beta}{720}\right)$$