6.012 - Electronic Devices and Circuits

Lecture 3 - Solving The Five Equations - Outline

- **Announcements**

  Handouts - 1. Lecture; 2. Photoconductivity; 3. Solving the 5 eqs.

  See website for Items 2 and 3.

- **Review**

  5 unknowns: $n(x,t)$, $p(x,t)$, $J_e(x,t)$, $J_h(x,t)$, $E(x,t)$

  5 equations: Gauss's law (1), Currents (2), Continuity (2)

  What isn't covered: Thermoelectric effects; Peltier cooling

- **Special cases we can solve (approximately) by hand**

  Carrier concentrations in uniformly doped material (Lect. 1)

  Uniform electric field in uniform material (drift) (Lect. 1)

  Low level uniform optical injection (LLI, τ_{min}) (Lect. 2)

  Photoconductivity (Lect. 2)

  Doping profile problems (depletion approximation) (Lects. 3,4)

  Non-uniform injection (QNR diffusion/flow) (Lect. 5)

- **Doping profile problems**

  Electrostatic potential

  Poisson's equation

Non-uniform doping/excitation: Summary

What we have so far:

Five things we care about (i.e. want to know):

Hole and electron concentrations: $p(x,t)$ and $n(x,t)$

Hole and electron currents: $J_{hx}(x,t)$ and $J_{ex}(x,t)$

Electric field: $E_x(x,t)$

And, amazingly, we already have five equations relating them:

Hole continuity: $$\frac{\partial p(x,t)}{\partial t} + \frac{1}{q}\frac{\partial J_h(x,t)}{\partial x} = G - R \approx G_{ext}(x,t) - \left[n(x,t)p(x,t) - n_i^2\right] r(t)$$

Electron continuity: $$\frac{\partial n(x,t)}{\partial t} - \frac{1}{q}\frac{\partial J_e(x,t)}{\partial x} = G - R \approx G_{ext}(x,t) - \left[n(x,t)p(x,t) - n_i^2\right] r(t)$$

Hole current density: $$J_h(x,t) = q\mu_h p(x,t)E(x,t) - qD_h \frac{\partial p(x,t)}{\partial x}$$

Electron current density: $$J_e(x,t) = q\mu_e n(x,t)E(x,t) + qD_e \frac{\partial n(x,t)}{\partial x}$$

Charge conservation: $$\rho(x,t) = \frac{\partial\left[\varepsilon(x)E_x(x,t)\right]}{\partial x} \approx q\left[p(x,t) - n(x,t) + N_d(x) - N_a(x)\right]$$

So...we're all set, right? No, and yes.....

We'll see today that it isn't easy to get a general solution, but we can prevail.

Thermoelectric effects* - the Seebeck and Peltier effects

(current fluxes caused by temperature gradients, and visa versa)

Hole current density, isothermal conditions:

Drift **Diffusion**

$$J_h = \mu_h p\left(-\frac{d[q\phi]}{dx}\right) + qD_h\left(-\frac{dp}{dx}\right)$$

Hole potential energy gradient — Concentration gradient

Hole current density, non-isothermal conditions:

Drift **Diffusion** **Seebeck Effect**

$$J_h = \mu_h p\left(-\frac{d[q\phi]}{dx}\right) + qD_h\left(-\frac{dp}{dx}\right) + qS_h p\left(-\frac{dT}{dx}\right)$$

Temperature gradient

Seebeck Effect: temperature gradient → current — Generator

Peltier Effect: current → temperature gradient — Cooler/heater

* A cultural item; we will only consider isothermal situations on 6.012 exams and problem sets.

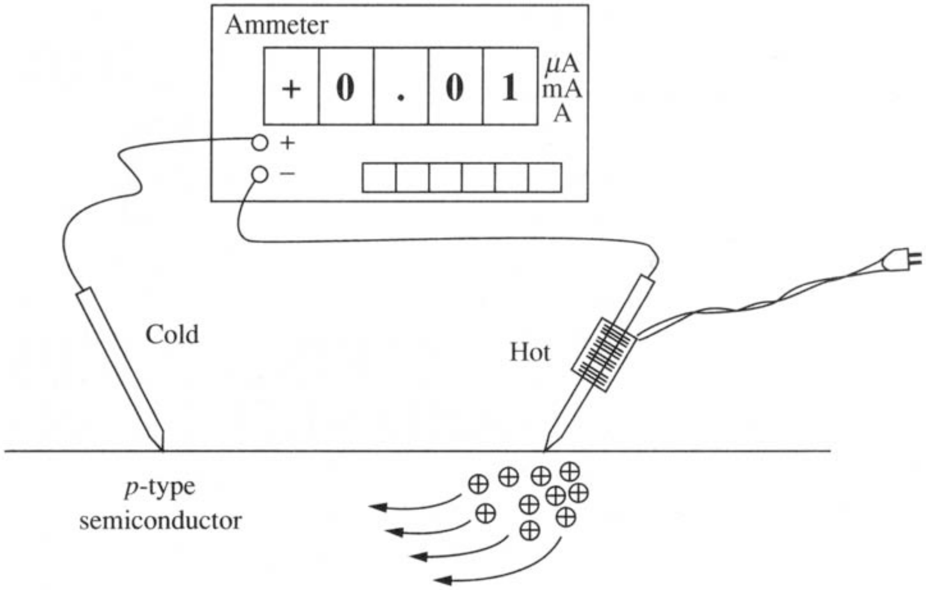
Thermoelectric effects - the Seebeck and Peltier effects

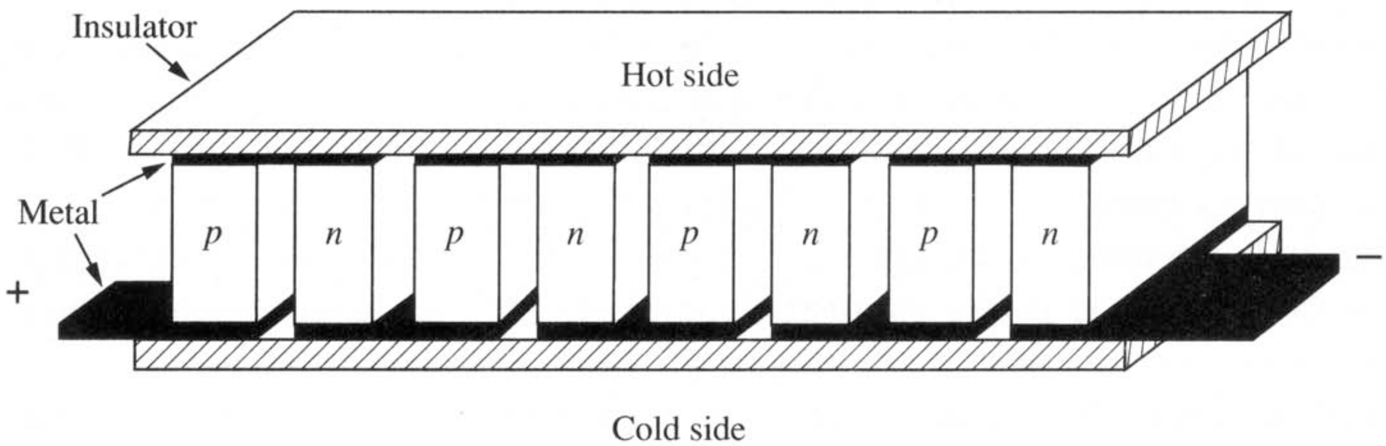
(current fluxes caused by temperature gradients, and visa versa)

Two examples:

Right - The hot point probe, an apparatus for determining the carrier type of semiconductor samples.

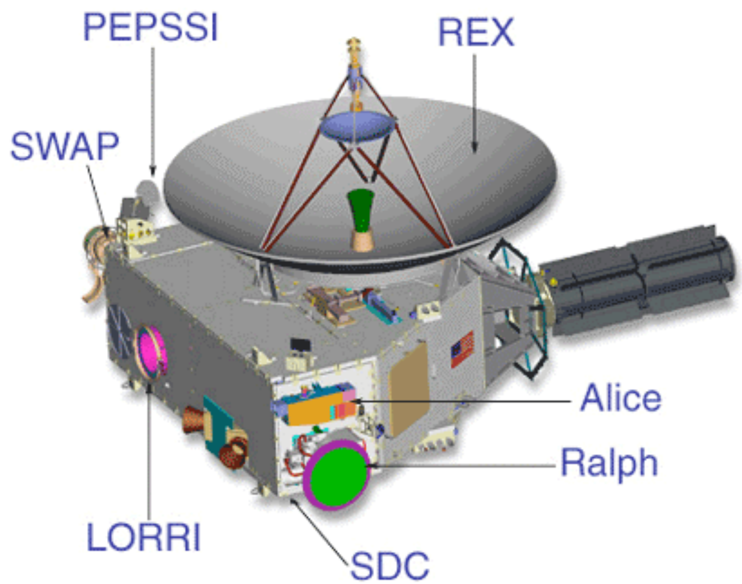
Below - A thermoelectric array like those in thermoelectric generators and solid-state refrigerators.





Thermoelectric Generators and Coolers -

Electrical power for a trip to Pluto



Source: NASA.

"...electrical power for the New Horizons spacecraft and science instruments [is] provided by a single radioisotope thermoelectric generator, or RTG."

Launched 1/19/2006

http://pluto.jhuapl.edu/

Source: NASA/Johns Hopkins University Applied Physics Laboratory/Southwest Research Institute.

Cooling/heating for the necessities of life

Image of thermoelectric wine cooler removed due to copyright restrictions.

Thermoelectric Wine Cooler

28 bottles

12˚C – 18˚C

Quiet, gas free, vibration free, environmentally friendly, LED display, interior light.

Zhongshan Candor Electric Appl. Co.

http://www.alibaba.com/

Non-uniform doping/excitation: Back to work

(laying the groundwork to model diodes and transistors)

What we have:

Five things we care about (i.e. want to know):

Hole and electron concentrations: $p(x,t)$ and $n(x,t)$

Hole and electron currents: $J_{hx}(x,t)$ and $J_{ex}(x,t)$

Electric field: $E_x(x,t)$

And, five equations relating them:

Hole continuity:
$$\frac{\partial p(x,t)}{\partial t} + \frac{1}{q}\frac{\partial J_h(x,t)}{\partial x} = G - R \approx G_{ext}(x,t) - \left[n(x,t)p(x,t) - n_i^2\right]r(t)$$

Electron continuity:
$$\frac{\partial n(x,t)}{\partial t} - \frac{1}{q}\frac{\partial J_e(x,t)}{\partial x} = G - R \approx G_{ext}(x,t) - \left[n(x,t)p(x,t) - n_i^2\right]r(t)$$

Hole current density:
$$J_h(x,t) = q\mu_h p(x,t)E(x,t) - qD_h \frac{\partial p(x,t)}{\partial x}$$

Electron current density:
$$J_e(x,t) = q\mu_e n(x,t)E(x,t) + qD_e \frac{\partial n(x,t)}{\partial x}$$

Charge conservation:
$$\rho(x,t) = \frac{\partial\left[\varepsilon(x)E_x(x,t)\right]}{\partial x} \approx q\left[p(x,t) - n(x,t) + N_d(x) - N_a(x)\right]$$

We can get approximate analytical solutions in 5 important cases!

Solving the five equations: special cases we can handle

1. Uniform doping, thermal equilibrium ($n_o p_o$ product, n_o, p_o):

$$\frac{\partial}{\partial x}=0, \quad \frac{\partial}{\partial t}=0, \quad g_L(x,t)=0, \quad J_e=J_h=0$$

Lecture 1

2. Uniform doping and E-field (drift conduction, Ohms law):

$$\frac{\partial}{\partial x}=0, \quad \frac{\partial}{\partial t}=0, \quad g_L(x,t)=0, \quad E_x \text{ constant}$$

Lecture 1

3. Uniform doping and uniform low level optical injection (τ_{min}):

$$\frac{\partial}{\partial x}=0, \quad g_L(t), \quad n' << p_o$$

Lecture 2

3'. Uniform doping, optical injection, <u>and</u> E-field (photoconductivity):

$$\frac{\partial}{\partial x}=0, \quad E_x \text{ constant}, \quad g_L(t)$$

Lecture 2

4. Non-uniform doping in thermal equilibrium (junctions, interfaces)

$$\frac{\partial}{\partial t}=0, \quad g_L(x,t)=0, \quad J_e=J_h=0$$

Lectures 3,4

5. Uniform doping, non-uniform LL injection (QNR diffusion)

$$\frac{\partial N_d}{\partial x}=\frac{\partial N_a}{\partial x}=0, \quad n' \approx \mathrm{p}', \quad \mathrm{n}' << \mathrm{p}_\mathrm{o}, \quad J_e \approx qD_e\frac{\partial n'}{\partial x}, \quad \frac{\partial}{\partial t}\approx 0$$

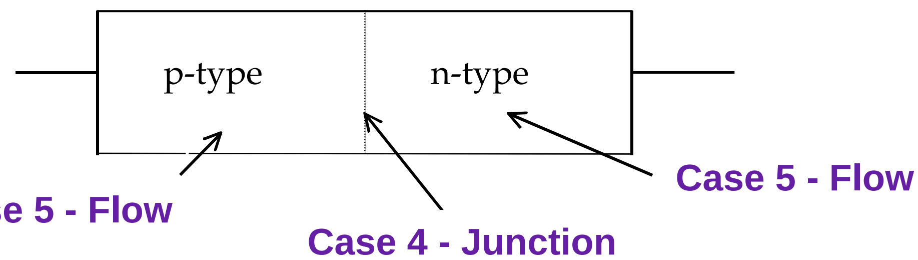
Lecture 5

Non-uniform material with non-uniform excitations

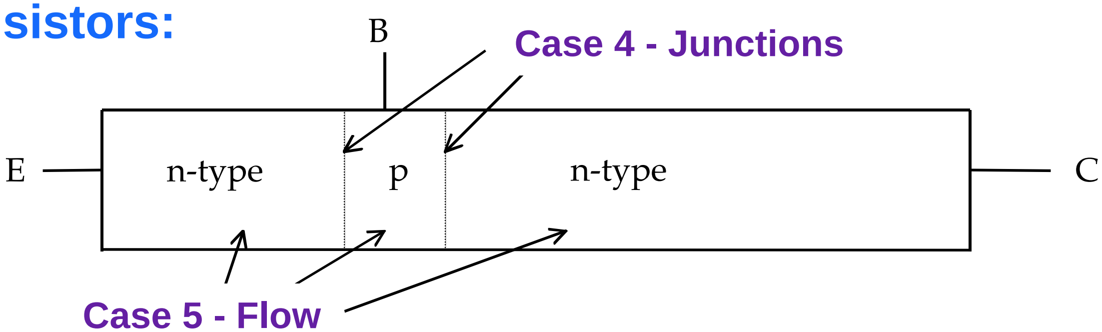
(laying the groundwork to model diodes and transistors)

Where cases 2, 4, and 5 appear in important semiconductor devices

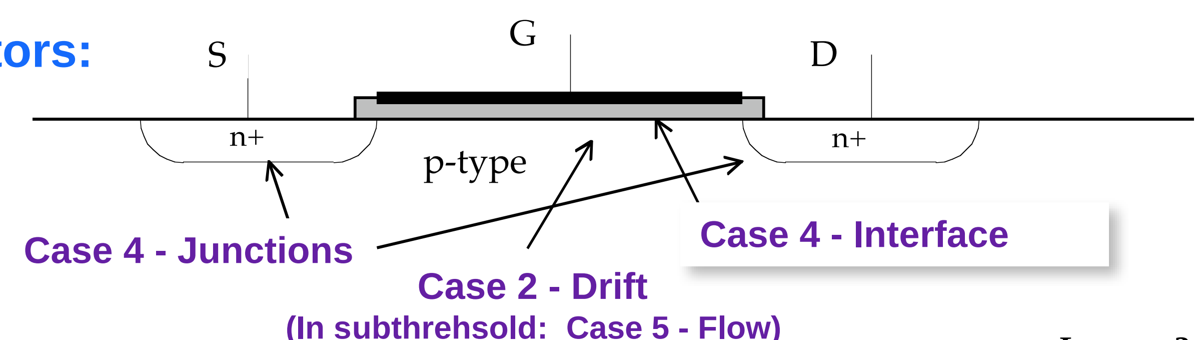
Junction diodes, LEDs:



Bipolar transistors:



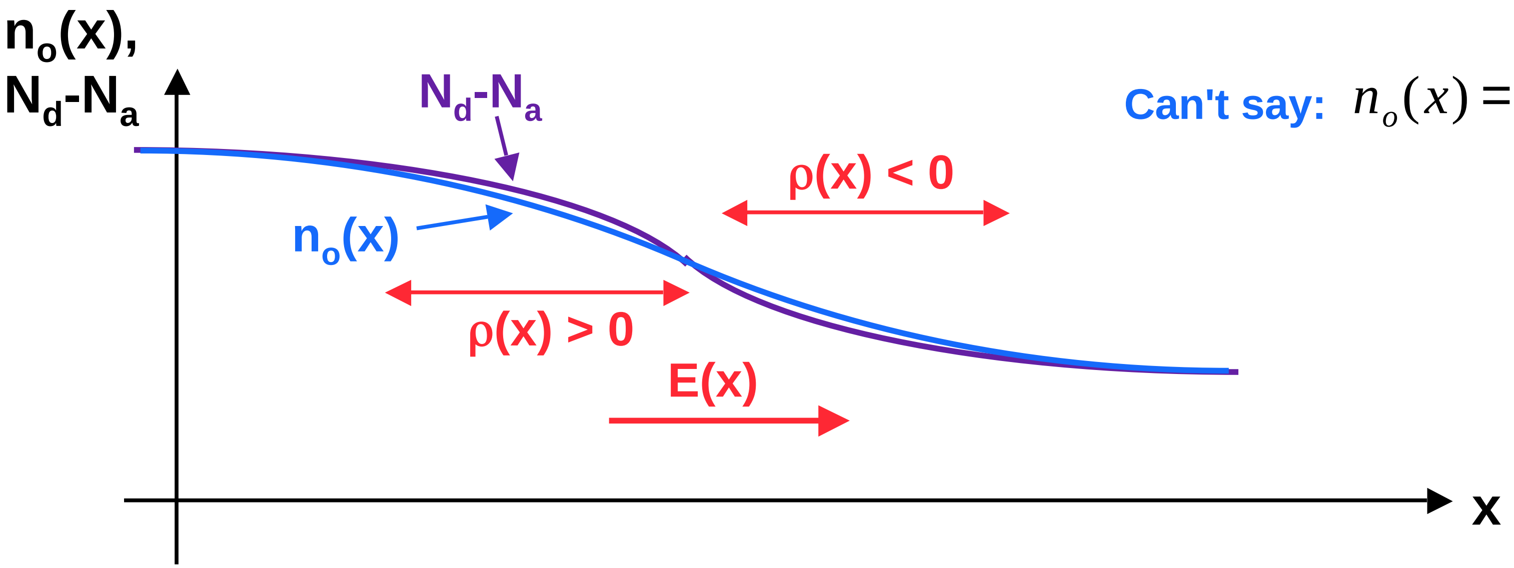
MOS transistors:



Case 4: Non-uniform doping in thermal equilibrium

Doping Profiles and p-n Junctions in TE: $N_a(x)$, $N_d(x)$

Any time the doping varies with position, we can no longer assume that there is charge neutrality everywhere and that $\rho(x) = 0$. The dopants are fixed, but the carriers are mobile and diffuse:



Can't say: $n_o(x) = N_D(x)$

Electron diffusion →
Electron drift ←
Hole drift →
Hole diffusion ←

In T.E., the net fluxes must be zero

Non-uniform doping in thermal equilibrium, cont.

To treat non-uniformly doped materials we begin by looking at them in <u>thermal equilibrium</u>, as we've said.

This is useful because in thermal equilibrium we must have:

$$g_L(x,t) = 0$$

$$n(x,t) = n_o(x)$$

$$p(x,t) = p_o(x)$$

$$J_e(x,t) = 0$$

$$J_h(x,t) = 0$$

Consequently, the 2 continuity equations in our 5 equations reduce to 0 = 0, e.g.:

$$\underbrace{\frac{\partial n(x,t)}{\partial t}}_{0} - \frac{1}{q}\underbrace{\frac{\partial J_e(x,t)}{\partial x}}_{0} = \underbrace{g_L(x,t)}_{0} - \left[\underbrace{n(x,t)\cdot p(x,t) - n_o(x)\cdot p_o(x)}_{0}\right] r(T)$$

Non-uniform doping in thermal equilibrium, cont.

The third and fourth equations, the current equations, give:

$$0 = q\mu_e n_o(x)E(x) + qD_e \frac{dn_o(x)}{dx} \quad \rightarrow \quad \frac{d\phi}{dx} = \frac{D_e}{\mu_e}\frac{1}{n_o(x)}\frac{dn_o(x)}{dx}$$

$$0 = q\mu_h p_o(x)E(x) - qD_h \frac{dp_o(x)}{dx} \quad \rightarrow \quad \frac{d\phi}{dx} = -\frac{D_h}{\mu_h}\frac{1}{p_o(x)}\frac{dp_o(x)}{dx}$$

And Poisson's equation becomes:

$$\frac{dE(x)}{dx} = -\frac{d^2\phi(x)}{dx^2} = \frac{q}{\varepsilon}\left[p_o(x) - n_o(x) + N_d(x) - N_a(x)\right]$$

In the end, we have <u>three equations</u> in our <u>three remaining unknowns</u>, $n_o(x)$, $p_o(x)$, and $\phi(x)$, so all is right with the world.

Non-uniform doping in thermal equilibrium, cont.

Looking initially at the first of our new set of equations, we note that both sides can be easily integrated with respect to position :

$$\int_{x_o}^{x} \frac{d\phi}{dx} dx = \frac{D_e}{\mu_e} \int_{x_o}^{x} \frac{1}{n_o(x)} \frac{dn_o(x)}{dx} dx$$

$$\phi(x) - \phi(x_o) = \frac{D_e}{\mu_e} \left[\ln n_o(x) - \ln n_o(x_o) \right] = \frac{D_e}{\mu_e} \ln \frac{n_o(x)}{n_o(x_o)}$$

Next, raising both sides to the e power yields:

$$n_o(x) = n_o(x_o) e^{\frac{\mu_e}{D_e}\left[\phi(x) - \phi(x_o)\right]}$$

We chose intrinsic material as our zero reference for the electrostatic potential:

$$\phi(x) = 0 \quad \text{where} \quad n_o(x) = n_i$$

and arrive at :

$$n_o(x) = n_i e^{\frac{\mu_e}{D_e}\phi(x)}$$

Non-uniform doping in thermal equilibrium, cont.

From the corresponding equation for holes we also find :

$$p_o(x) = n_i e^{-\frac{\mu_h}{D_h}\phi(x)}$$

Next use the Einstein relation: $$\frac{\mu_h}{D_h} = \frac{\mu_e}{D_e} = \frac{q}{kT}$$

Incredibly Multilingually rhyming

Note: this relationship rhymes as written, as well as when inverted, and also either way in Spanish. It is a very fundamental, and important, relationship!

$$\text{Note: } @\ \text{R.T.} \quad q/kT \approx 40\, V^{-1} \ \text{ and } \ kT/q \approx 25\, mV$$

Using the Einstein relation we have:

$$\boxed{n_o(x) = n_i e^{q\phi(x)/kT}} \quad \text{and} \quad \boxed{p_o(x) = n_i e^{-q\phi(x)/kT}}$$

Finally, putting these in Poisson's equation, a single equation for ϕ(x) in a doped semiconductor in TE materializes:

$$\boxed{\frac{d^2\phi(x)}{dx^2} = -\frac{q}{\varepsilon}\left[n_i\left(e^{-q\phi(x)/kT} - e^{q\phi(x)/kT}\right) + N_d(x) - N_a(x)\right]}$$

Non-uniform doping in thermal equilibrium, cont.

(an aside)

What do these equations say?

$$n_o(x) = n_i e^{q\phi(x)/kT} \quad \text{and} \quad p_o(x) = n_i e^{-q\phi(x)/kT}$$

To see, consider what they tell us about the ratio of the hole concentration at x_2, where the electrostatic potential is ϕ_2, and that at x_1, ϕ_1:

$$p_o(x_2) = p_o(x_1) e^{-q[\phi(x_2) - \phi(x_1)]/kT}$$

The thermal energy is kT, and the change in potential energy of a hole moved from x_1 to x_2 is $q(\phi_2 - \phi_1)$, so have:

$$\boxed{p_o(x_2) = p_o(x_1) e^{-\Delta PE_{x_1 \rightarrow x_2}/kT}}$$

If the potential energy is higher at x_2, than at x_1, then the population is lower at x_2 by a factor $e^{-\Delta PE/kT}$.

That is, the population is lower at the top of a potential hill.

If the potential energy is lower, then the population is higher.

That is, the population is, conversely, higher at the bottom of a potential hill.

Non-uniform doping in thermal equilibrium, cont.

(continuing the aside)

The factor $e^{-\Delta PE/kT}$ is called a Boltzman factor. It is a factor relating the population densities of particles in many situations, such as gas molecules in an ideal gas under the influence of gravity (i.e, the air above the surface of the earth) and conduction electrons and holes in a semiconductor.*

The potential energy difference for holes is $q\Delta\phi$, while that for electrons is $-q\Delta\phi$. Thus when we look at the electron and hole populations at a point where the electrostatic potential is ϕ, relative to those where the potential is zero (and both populations are n_i) we have:

$$n_o(x) = n_i e^{q\phi(x)/kT} \quad \text{and} \quad p_o(x) = n_i e^{-q\phi(x)/kT}$$

We will return to this picture of populations on either side of a potential hill when we examine at the minority carrier populations on either side of a biased p-n junction.

*** Until the doping levels are very high, in which case the Boltzman factor must be replaced by a Fermi factor.****

**** Don't worry about it.**

Doing the numbers:

I. D to μ conversions, and visa versa

To convert between D and μ it is convenient to say kT/q ≈ 25 mV, in which case q/kT ≈ 40 V^{-1}:

17˚C/62˚F

Example 1: **μ_e = 1600 cm^2/V-s, μ_h = 600 cm^2/V-s**

$$D_e = \mu_e / (q/kT) = 1600/40 = 40\ cm^2/s$$

$$D_h = \mu_h / (q/kT) = 600/40 = 15\ cm^2/s$$

II. Relating ϕ to n and p, and visa versa

To calculate ϕ knowing n or p it is better to say that kT/q ≈ 26 mV, because then (kT/q)ln10 ≈ 60 mV:

28˚C/83˚F

Example 1: **n-type, $N_D = N_d - N_a = 10^{16}\ cm^{-3}$**

$$\phi_n = \frac{kT}{q}\ln\frac{10^{16}}{10^{10}} = \frac{kT}{q}\ln 10^6 = \frac{kT}{q}\ln 10 \cdot \log 10^6 \approx 0.06 \ln 10^6 = 0.36\ eV$$

Example 2: **p-type, $N_A = N_a - N_d = 10^{17}\ cm^{-3}$**

$$\phi_p = -\frac{kT}{q}\ln\frac{10^{17}}{10^{10}} = -\frac{kT}{q}\ln 10 \cdot \log 10^7 \approx -0.06 \cdot 7 = -0.42\ eV$$

Example 3: **60 mV rule:**

For every order of magnitude the doping is above (below) n_i, the potential increases (decreases) by 60 meV.

More numbers

	n_o [cm^{-3}]	p_o [cm^{-3}]	ϕ [V]	
	10^{19}	10^{1}	0.54	
	10^{18}	10^{2}	0.48	
	10^{17}	10^{3}	0.42	Typical range
n-type	10^{16}	10^{4}	0.36	
	10^{15}	10^{5}	0.30	
	10^{14}	10^{6}	0.24	
	10^{13}	10^{7}	0.18	
	10^{12}	10^{8}	0.12	
	10^{11}	10^{9}	0.06	
Intrinsic →	10^{10}	10^{10}	0.00	
	10^{9}	10^{11}	-0.06	
	10^{8}	10^{12}	-0.12	
	10^{7}	10^{13}	-0.18	
	10^{6}	10^{14}	-0.24	
p-type	10^{5}	10^{15}	-0.30	
	10^{4}	10^{16}	-0.36	
	10^{3}	10^{17}	-0.42	Typical range
	10^{2}	10^{18}	-0.48	
	10^{1}	10^{19}	-0.54	

Non-uniform doping in thermal equilibrium,cont:

We have reduced our problem to solving one equation for one unknown, in this case $\phi(x)$:

$$\frac{d^2\phi(x)}{dx^2} = -\frac{q}{\varepsilon}\left[n_i\left(e^{-q\phi(x)/kT} - e^{q\phi(x)/kT}\right) + N_d(x) - N_a(x)\right]$$

Once we find $\phi(x)$ we can find n_o and p_o from:

$$n_o(x) = n_i e^{q\phi(x)/kT} \quad \text{and} \quad p_o(x) = n_i e^{-q\phi(x)/kT}$$

Solving Poisson's equation for $\phi(x)$ is in general non-trivial, and for precise answers a "Poisson Solver" program must be employed. However, in two special cases we can find very useful, insightful approximate analytical solutions:

Case I: Abrupt changes from p- to n-type (i.e., junctions)

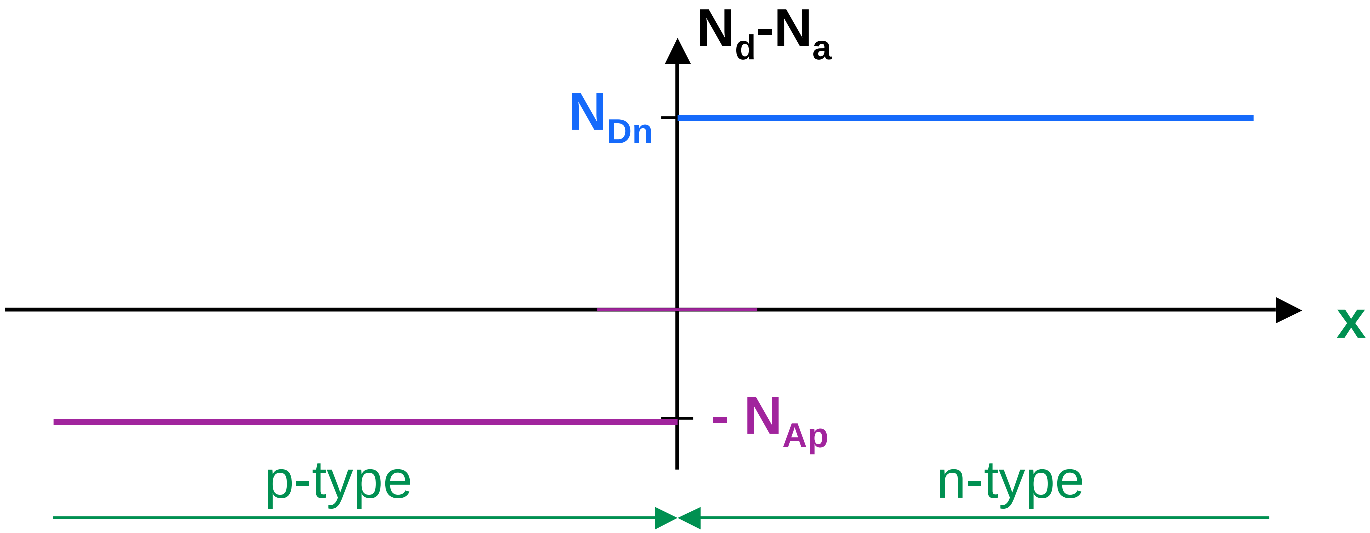
also: surfaces (Si to air or other insulator)
interfaces (Si to metal, Si to insulator, or Si to insulator to metal)

Case II: Slowly varying doping profiles.

Non-uniform doping in thermal equilibrium, cont.:

Case I: Abrupt p-n junctions

Consider the profile below:



$$p_o = N_{Ap}, \quad n_o = n_i^2 / N_{Ap}$$

$$\phi = -\frac{kT}{q} \ln\left(N_{Ap}/n_i\right) \equiv \phi_p$$

?

$$n_o(x) = ?$$

$$p_o(x) = ?$$

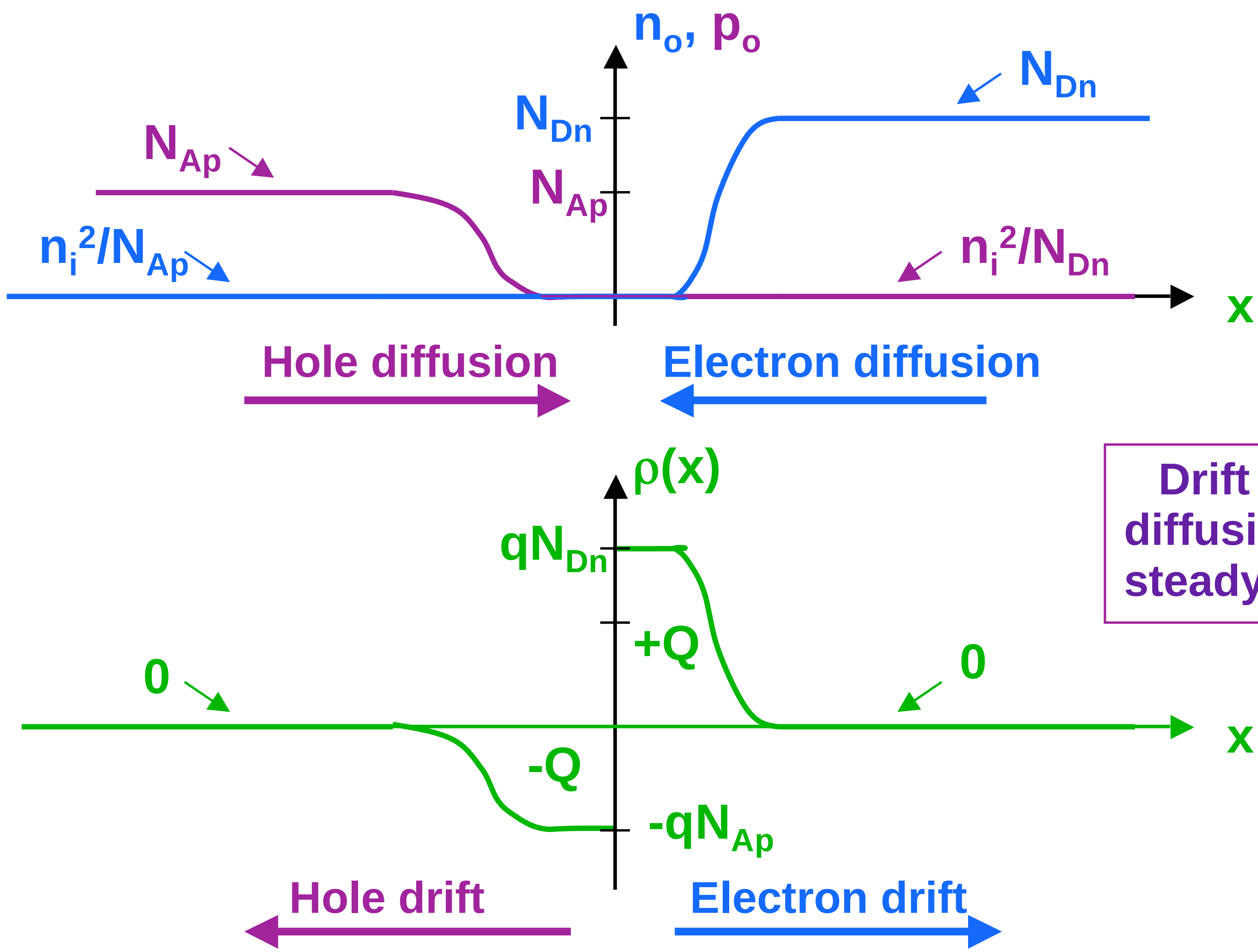
$$\phi(x) = ?$$

$$n_o = N_{Dn}, \quad p_o = n_i^2 / N_{Dn}$$

$$\phi = \frac{kT}{q} \ln\left(N_{Dn}/n_i\right) \equiv \phi_n$$

Abrupt p-n junctions, cont:

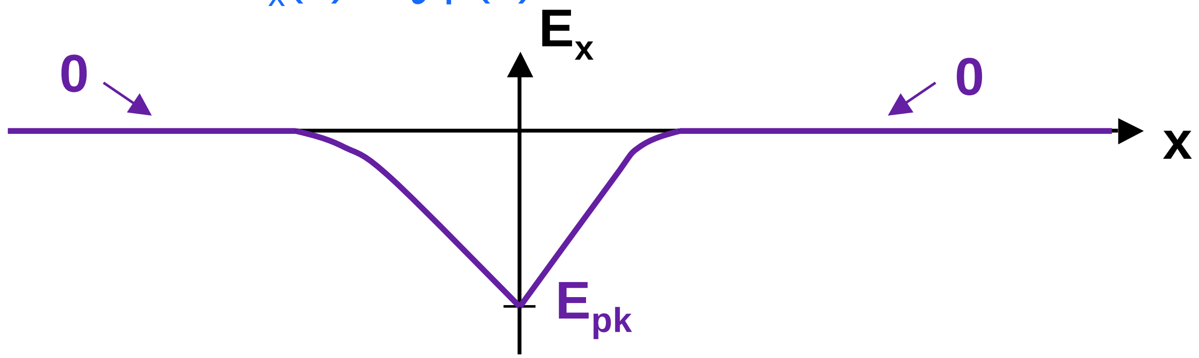
First look why there is a dipole layer in the vicinity of the junction, and a "built-in" electric field.



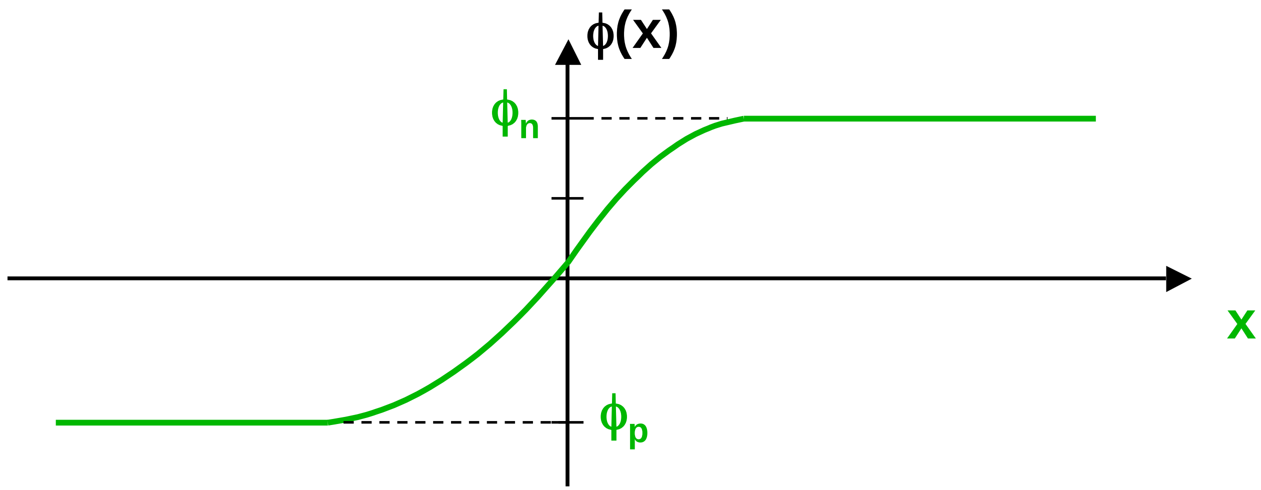
Drift balances diffusion in the steady state.

Abrupt p-n junctions, cont:

If the charge density is no longer zero there must be an electric field: $\varepsilon E_x(x) = \int \rho(x)dx$



and an electrostatic potential step: $\phi(x) = -\int E_x(x)dx$



Ok, but how do we find $\phi(x)$?

Abrupt p-n Junctions: *the general strategy*

We have to solve an non-linear, second order differential equation for ϕ:

$$\frac{d^2\phi(x)}{dx^2} = -\frac{\rho(x)}{\varepsilon} = -\frac{q}{\varepsilon}\left[n_i\left(e^{-q\phi(x)/kT} - e^{q\phi(x)/kT}\right) + N_d(x) - N_a(x)\right]$$

Or, alternatively $\qquad \phi(x) = -\iint \frac{\rho(x)}{\varepsilon}\,dx + Ax^2 + Bx$

In the case of an abrupt p-n junction we have a pretty good idea of what $\rho(x)$ must look like, and we know the details will be lost anyway after integrating twice, so we can try the following iteration strategy:

Guess a starting $\rho(x)$.

Integrated once to get E(x), and again to get $\phi(x)$.

Use $\phi(x)$ to find $p_o(x)$, $n_o(x)$, and, ultimately, a new $\rho(x)$.

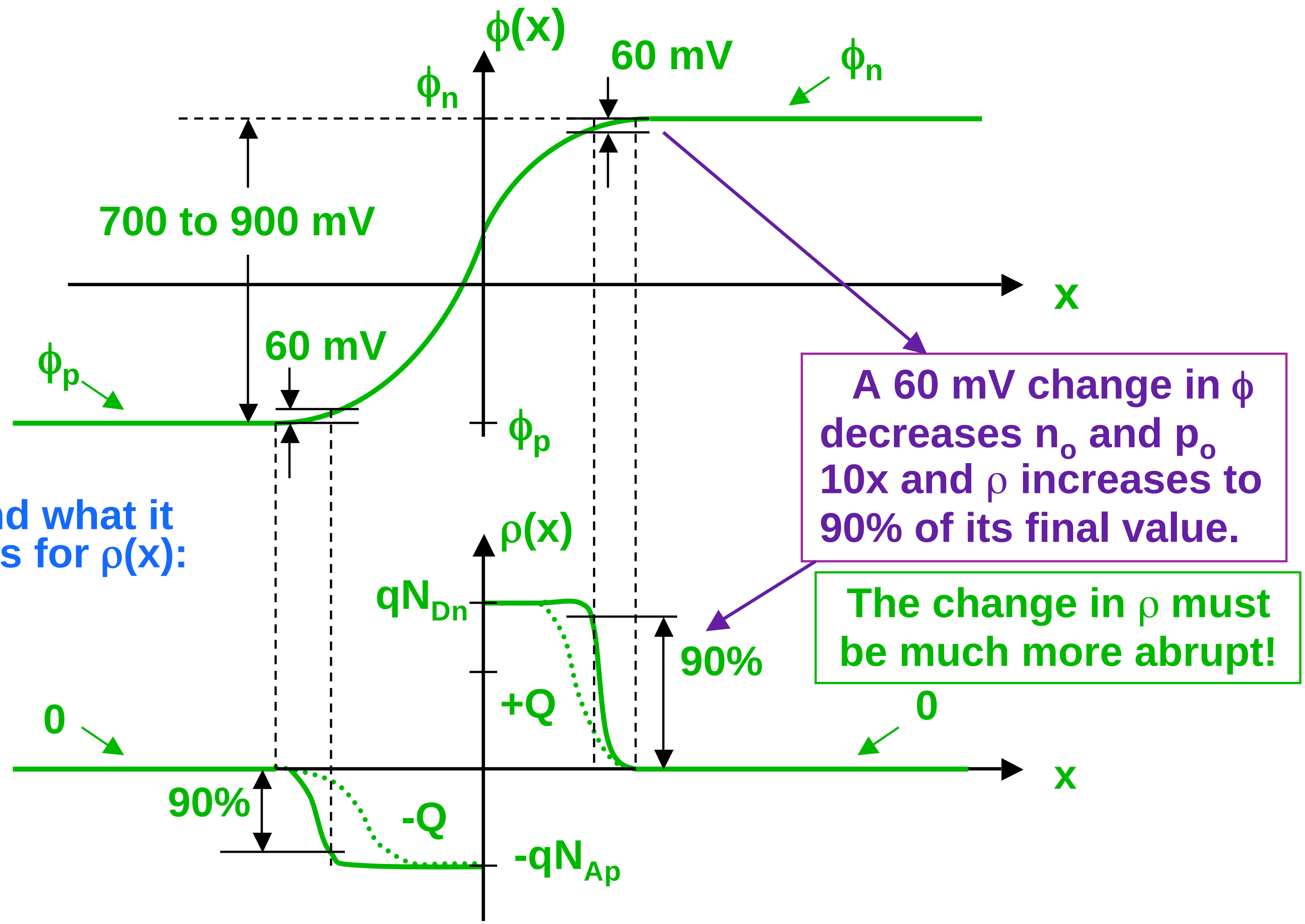
Compare the new $\rho(x)$ to the starting $\rho(x)$.

- If it is not close enough, use the new $\rho(x)$ to iterate again.

- If it is close enough, quit.

To figure out a good first guess for $\rho(x)$, look at how quickly n_o and p_o must change by looking first at how ϕ changes:

…and what it means for $\rho(x)$:



The observation that ρ changes a lot, when ϕ changes a little, is the key to the depletion approximation.

Lecture 3 - Solving The Five Equations - Summary

- **Non-uniform excitation in non-uniform samples**
  - **The 5 unkowns:** $n(x,t)$, $p(x,t)$, $J_e(x,t)$, $J_h(x,t)$, $E(x,t)$
  - **The 5 equations:** coupled, non-linear differential equations
- **Special cases we can solve (approximately)**
  - **Carrier concentrations:** **(Lect. 1)**
  - **Drift:** $J_{drift} = J_{e,drift} + J_{h,drift} = q\,(\mu_e\, n_o + \mu_h\, p_o)\, E = \sigma_o\, E$ **(Lect. 2)**
  - **Low level optical injection:** $dn'/dt - n'/\tau_{min} \approx g_L(t)$ **(Lect. 2)**
  - **Doping profile problems:** junctions and interfaces
  - **Non-uniform injection:** QNR flow problems
- **Using the hand solutions to model devices**
  - **pn Diodes:** two flow problems and a depl. approx.
  - **BJTs:** three flow problems and two depl. approx.'s
  - **MOSFETs:** three depl. approx.'s and one drift
- **Non-uniform doping in T.E.**
  - **Relating n_o, p_o, and electrostatic potential, ϕ**
  - **Poisson's equation:** two situations important in devices