Massachusetts Institute of Technology

Department of Electrical Engineering and Computer Science

6.011: Introduction to Communication, Control and Signal Processing

FINAL EXAM, May 18, 2010

ANSWER BOOKLET

Your Full Name:	SOLUTIONS
Recitation Time :	o'clock

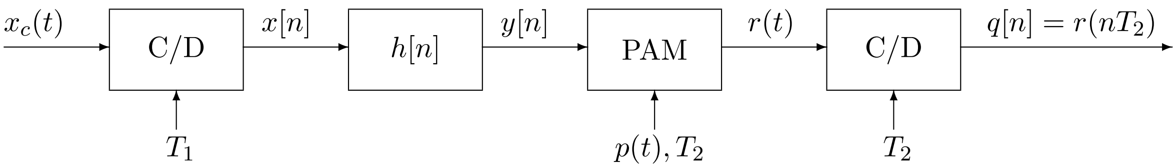
- This exam is **closed book**, but **4** sheets of notes are allowed. Calculators and other electronic aids will not be necessary and are not allowed.
- Check that this **ANSWER BOOKLET** has pages numbered up to 26. The booklet contains spaces for **all** relevant reasoning and answers.
- **Neat work and clear explanations count; show all relevant work and reasoning!** You may want to first work things through on scratch paper and then neatly transfer to this booklet the work you would like us to look at. Let us know if you need additional scratch paper. **Only** this booklet will be considered in the grading; **no additional answer or solution written elsewhere will be considered.** Absolutely no exceptions!
- There are **5 problems, weighted as shown, for a total of 100 points**. (The points indicated on the following pages for the various subparts of the problems are our best guess for now, but may be modified slightly when we get to grading.)

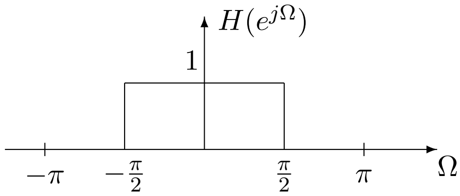
Problem	**Your Score**
1 (17 points)	
2 (18 points)	
3 (25 points)	
4 (20 points)	
5 (20 points)	
Total (100 points)	

Problem 1 (17 points)

Note that 1(d) does not depend on your answers to 1(a)-(c), and can be done independently of them.

$$X_c(j\omega) = 0 \quad \text{for} \quad |\omega| \geq 2\pi \times 10^3 \ .$$





1(a) (3 points) Determine the *largest* value of T_1 to ensure that $y[n] = x_c(nT_1)$.

Solution: For $y[n] = x_c(nT_1) = x[n]$, we need to choose T_1 so that $X(e^{j\Omega})$ is bandlimited to $\pi/2$. You might anticipate that this will require sampling at *twice* the Nyquist rate. After sampling (and assuming no aliasing — a good assumption if we will be sampling at twice the Nyquist rate), we know $X(e^{j\Omega})$ is related to its continuous counterpart as follows in the interval $|\Omega| < \pi$ (and repeats periodically with period 2π outside this interval):

$$X(e^{j\Omega}) = \frac{1}{T_1} X_c(j\omega)\Big|_{\omega=\frac{\Omega}{T_1}} \ .$$

Since $X_c(j\omega)$ is bandlimited to $2\pi \times 10^3$, we know from the above relation that the highest frequency in $X(e^{j\Omega})$ is $\Omega_o = (2\pi \times 10^3)T_1$. Since we require $|\Omega_o| \leq \pi/2$, we conclude that $T_1 \leq 2.5 \times 10^{-4}$. This does indeed correspond to sampling at twice the Nyquist rate for the signal $x_c(t)$.

1(b) (6 points) With T_1 picked as in 1(a), determine a choice for T_2 and $p(t)$ to ensure that

$$r(t) = x_c(t) \ .$$

(You can leave your expressions for T_2 and $p(t)$ in terms of T_1, instead of substituting in the numerical value you obtained in 1(a) for T_1.)

Solution: For $r(t) = x_c(t)$, with the PAM relation

$$r(t) = \sum_n y[n]p(t - nT_2) \ ,$$

we need to perfectly reconstruct $x_c(t)$ from the samples $y[n]$. Since we chose T_1 in part 1(a) to ensure that $y[n] = x[n]$, attaining the equality $r(t) = x_c(t)$ amounts to using the PAM block in the figure to perfectly reconstruct $x_c(t)$ from the samples $y[n] = x[n] = x(nT_1)$. This is possible, by the sampling theorem, since the samples are collected at greater than the Nyquist rate. Specifically, perfect reconstruction can be achieved by interpolating the samples $y[n]$ with interval T_1 using a sinc function (i.e., using an ideal D/C converter), as follows:

$$r(t) = \sum_n y[n] \frac{\sin(\frac{\pi}{T_1}(t - nT_1))}{\frac{\pi}{T_1}(t - nT_1)} \ .$$

(This is the standard sampling-theorem reconstruction of a bandlimited signal from samples taken at a sufficiently high rate.) Hence we can choose $T_2 = T_1$ and use the PAM pulse

$$p(t) = \frac{\sin(\frac{\pi}{T_1}t)}{\frac{\pi}{T_1}t} \ .$$

Note for future reference that $P(j\omega) = T_1$ in the interval $|\omega| < (\pi/T_1)$, and 0 elsewhere.

(**Continue 1(b) on next page**:)

1(b) (continued) Also determine if there is another choice of T_2 and $p(t)$ that could ensure the equality $r(t) = x_c(t)$. Explain your answer carefully.

Solution: Since our samples were obtained at twice the Nyquist rate, we can actually use any interpolating pulse whose transform $P(j\omega) = T_1$ in the interval $|\omega| < \pi/(2T_1)$, where $X(e^{j\Omega})\Big|_{\Omega=\omega T_1}$ is nonzero, and 0 wherever else $X(e^{j\Omega})\Big|_{\Omega=\omega T_1}$ is nonzero; the choice of $P(j\omega)$ in intervals where $X(e^{j\Omega})\Big|_{\Omega=\omega T_1} = 0$ is arbitrary. To see this more concretely, note that the PAM relation

$$r(t) = \sum_n y[n]p(t - nT_2)$$

translates in the frequency domain to

$$\begin{aligned}
R(j\omega) &= \sum_n y[n]e^{-j\omega nT_2}P(j\omega) \\
&= \left(\sum_n y[n]e^{-j\omega nT_2}\right)P(j\omega) \\
&= Y(e^{j\Omega})\Big|_{\Omega=\omega T_2}P(j\omega)
\end{aligned}$$

We chose T_1 in part 1(a) to ensure that $y[n] = x(nT_1) = x[n]$, so we can write

$$R(j\omega) = X(e^{j\Omega})\Big|_{\Omega=\omega T_2}P(j\omega)\ .$$

To get $r(t) = x_c(t)$, we must ensure $R(j\omega) = X_c(j\omega)$, so we require

$$X(e^{j\Omega})\Big|_{\Omega=\omega T_2}P(j\omega) = X(e^{j\Omega})\Big|_{\Omega=\omega T_1}T_1$$

for $|\omega| < \pi/(2T_1)$, and require the left side to be 0 for $|\omega| > \pi/(2T_1)$.

The above expression implies the constraints $T_2 = T_1$ and $P(j\omega) = T_1$ for $|\omega| \le \frac{\pi}{2T_1}$. Moving to higher values of $|\omega|$, note that since $X(e^{j\Omega})|_{\Omega=\omega T_1}$ replicates at multiples of $\frac{2\pi}{T_1}$, we need $P(j\omega)$ to be zero before the first replication edge of $X(e^{j\Omega})\Big|_{\Omega=\omega T_1}$ at $\frac{2\pi}{T_1} - \frac{\pi}{2T_1} = \frac{3\pi}{2T_1}$. (One can have $P(j\omega)$ change back to nonzero values at yet higher $|\omega|$ for which $X(e^{j\Omega})\Big|_{\Omega=\omega/T_1}$ goes to 0 again, but we don't pursue this here.)

If we stick to sinc functions for $p(t)$, then we achieve $r(t) = x_c(t)$ if we use a PAM block with $T_2 = T_1$ and

$$p(t) = \frac{\sin(\frac{\pi}{T_3}t)}{\frac{\pi}{T_1}t}\ ,$$

where

$$\frac{\pi}{2T_1} \leq \frac{\pi}{T_3} \leq \frac{3\pi}{2T_1} \ ,$$

or

$$\frac{2T_1}{3} \leq T_3 \leq 2T_1 \ .$$

1(c) (3 points) With T_1 picked as in 1(a), how would you modify your choice of T_2 and $p(t)$ from 1(b) to ensure that

$$r(t) = x_c(2.7t) \ .$$

Solution: From part (b) we know one choice of PAM parameters that ensures $r(t) = x_c(t)$ is $T_2 = T_1$ and $p(t) = \frac{\sin(\frac{\pi}{T_2}t)}{\frac{\pi}{T_2}t}$. This allows us to express $r(t) = x_c(t)$ as

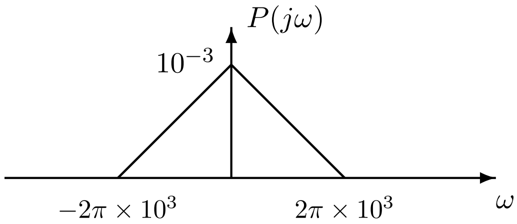
$$r(t) = x_c(t) = \sum_n y[n] \frac{\sin(\pi(\frac{t}{T_1} - n))}{\pi(\frac{t}{T_1} - n)}$$

For $r(t)$ to equal $x_c(2.7t)$, we subsitute $2.7t$ for t in the above relation to get

$$\begin{aligned} r(t) = x_c(2.7t) &= \sum_n y[n] \frac{\sin(\pi(\frac{2.7t}{T_1} - n))}{\pi(\frac{2.7t}{T_1} - n)} \\ &= \sum_n y[n] \frac{\sin(\pi(\frac{t}{T_2} - n))}{\pi(\frac{t}{T_2} - n)} \end{aligned}$$

where $T_2 = T_1/2.7$. So, a PAM block with $T_2 = T_1/2.7$ and $p(t) = \frac{\sin(\frac{\pi t}{T_2})}{\frac{\pi t}{T_2}}$ will ensure that $r(t) = x_c(2.7t)$.

1(d) (4 points) Assume that $p(t)$ is now chosen so that its CTFT, $P(j\omega)$, is as shown below. Determine a value of T_2 to ensure that $q[n] = y[n]$.



Solution: To ensure that $q[n] = y[n]$, we must choose T_2 so that the pulse $P(j\omega)$ satisfies the Nyquist condition for zero ISI. In the frequency domain, this condition requires that the sum of replicas of $P(j\omega)$ centered at integer multiples of $\frac{2\pi}{T_2}$ adds up to T_2:

$$T_2 = \sum_k P(j(\omega - \frac{2\pi}{T_2}k))$$

For the given pulse $P(j\omega)$, choosing $\frac{2\pi}{T_2} = 2\pi \times 10^3$ or $T_2 = 10^{-3}$ allows us to satisfy the Nyquist condition for zero ISI. Note that this T_2 is four times the T_2 picked in (b), but this is because now we are only trying to preserve the samples, not the CT waveform.

Problem 2 (18 points)

For each of the following parts, write down whether the statement is **True** or **False** (circle whichever is appropriate), giving a clear explanation or counterexample. (Take care with this!)

2(a) (4 points) Suppose $x[n]$ is a zero-mean discrete-time (DT) wide-sense stationary (WSS) random process. If its autocorrelation function $R_{xx}[m]$ is 0 for $|m| \geq 2$ but nonzero for $m = -1, 0, 1$, then the linear minimum mean-square-error (LMMSE) estimator of $x[n+1]$ from measurements of $x[n]$ and $x[n-1]$, namely

$$\widehat{x}[n+1] = a_0 x[n] + a_1 x[n-1] \ ,$$

will necessarily have $a_1 = 0$.

TRUE **FALSE**

Explanation/counterexample:

Solution: FALSE.

Since the mean of $x[n]$ is zero, $R_{xx}[m] = C_{xx}[m]$. The coefficients a_0 and a_1 are formed by solving the normal equations:

$$\begin{bmatrix} C_{xx}[0] & C_{xx}[1] \\ C_{xx}[-1] & C_{xx}[0] \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} C_{xx}[-1] \\ 0 \end{bmatrix}$$

This matrix equation cannot be solved with $a_1 = 0$.

2(b) (4 points) If the power spectral density $S_{yy}(j\omega)$ of a continuous-time (CT) WSS random process $y(t)$ is given by

$$S_{yy}(j\omega) = \frac{17+\omega^2}{23+\omega^2}$$

then the mean value of the process is zero, i.e., $\mu_y = E[y(t)] = 0$.

TRUE **FALSE**

Explanation/counterexample:

Solution: TRUE.

In general, $C_{yy}(\tau) \leftrightarrow D_{yy}(j\omega) = S_{yy}(j\omega) - \mu_y^2 2\pi\delta(\omega)$, and $D_{yy}(j\omega) \geq 0$. For the given $S_{yy}(j\omega)$, the inequality $D_{yy}(j\omega) \geq 0$ cannot be satisfied unless $\mu_y = 0$ since $S_{yy}(j\omega)$ has no impulses at $\omega = 0$.

2(c) (4 points) If the autocovariance function $C_{vv}[m]$ of a DT WSS random process $v[n]$ is given by

$$C_{vv}[m] = \left(\frac{1}{3}\right)^{|m|} ,$$

then the LMMSE estimator of $v[n+1]$ from all past measurements, which we write as

$$\widehat{v}[n+1] = \left(\sum_{k=0}^{\infty} h_k v[n-k]\right) + d ,$$

will have $h_k = 0$ for all $k \geq 1$, i.e., only the coefficients h_0 and d can be nonzero.

TRUE **FALSE**

Explanation/counterexample:

Solution: TRUE.

The claim is that the LMMSE estimator of $v[n+1]$ has the form:

$$\widehat{v}[n+1] = h_0 v[n] + d$$

To verify this claim, we need to show that the estimation error is orthogonal to 1 (i.e., that the estimator is unbiased) and to $v[n]$, for some appropriately chosen h_0 and d.

Orthogonality to $v[n]$ implies:

$$\begin{aligned}
E[(v[n+1] - \widehat{v}[n+1])v[n]] &= 0 \\
R_{vv}[1] - h_0 R_{vv}[0] - d\mu_v &= 0 \\
C_{vv}[1] - h_0 C_{vv}[0] + (1-h_0)\mu_v^2 - d\mu_v &= 0 \\
\frac{1}{3} - h_0 + (1-h_0)\mu_v^2 - d\mu_v &= 0
\end{aligned}$$

Orthogonality to 1 implies:

$$\begin{aligned}
E[(v[n+1] - \widehat{v}[n+1])] &= 0 \\
\mu_v - h_0\mu_v - d &= 0
\end{aligned}$$

These equalities are satisfied when $h_0 = \frac{1}{3}$ and $d = \frac{2}{3}\mu_v$. Hence, $\widehat{v}[n+1] = \frac{1}{3}v[n] + \frac{2}{3}\mu_v$ is the LMMSE estimator of $v[n+1]$.

2(d) (3 points) The process $v[n]$ in 2(c) is ergodic in mean value.

TRUE **FALSE**

Explanation/counterexample:

Solution: TRUE.

Since the covariance function $C_{xx}[m]$ goes to zeros as $m \to \infty$, we can conclude that the process $v[n]$ is ergodic in the mean.

2(e) (3 points) If $z[n] = v[n] + W$, where $v[n]$ is the process in 2(c), and where W is a random variable with mean 0 and variance $\sigma_W^2 > 0$, then the process $z[n]$ is ergodic in mean value.

TRUE **FALSE**

Explanation/counterexample:

Solution: FALSE.

The ensemble average of $z[n]$ is $E[z[n]] = \mu_v$. However, the time-average of any particular realization of $z[n]$ will be $\mu_v + w$, where w is the particular realization of W in that outcome. Since the time-average of $z[n]$ does not equal its ensemble average, we conclude that the process $z[n]$ is not ergodic in the mean.

Problem 3 (25 points)

$$\begin{aligned} \mathbf{q}[n+1] &= \mathbf{A}\mathbf{q}[n] + \mathbf{b}x[n] + \mathbf{h}w[n] \ , \\ y[n] &= \mathbf{c}^T\mathbf{q}[n] + v[n] \ . \end{aligned}$$

where

$$\mathbf{q}[n] = \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} \frac{1}{2} & \frac{3}{4} \\ 0 & 2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}, \quad \mathbf{h} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{c}^T = \begin{bmatrix} 0 & 1 \end{bmatrix},$$

3(a) Determine the two natural frequencies of the system (i.e., the eigenvalues of $\mathbf{A}$), and for each of them specify whether the associated mode satisfies the properties listed on the next page.

Solution:.

The matrix $\mathbf{A}$ is upper triangular, so its eigenvalues are its diagonal entries. Hence, $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = 2$. The eigenvector associated with λ_1 is $\mathbf{v_1} = [1 \ 0]^T$ and the eigenvector associated with λ_2 is $\mathbf{v_2} = [\frac{1}{2} \ 1]^T$.

(**Write your answers on the next page.**)

3(a) (continued)(8 points)

The two eigenvalues are: $\lambda_1 = \frac{1}{2}$ $\lambda_2 = 2$

List below whichever of the eigenvalues, if either, has an associated mode that satisfies the indicated condition:

(i) decays asymptotically to 0 in the zero-input response:
λ_1 since $|\lambda_1| < 1$

(ii) is reachable from the input $x[n]$ (with $w[n]$ kept at zero):
λ_2 since $\mathbf{V}^{-1}\mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

(iii) is reachable from the input $w[n]$ (with $x[n]$ kept at zero):
λ_1 and λ_2 since $\mathbf{V}^{-1}\mathbf{h} = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$

(iv) is observable from the output $y[n]$:
λ_2 since $\mathbf{c}^T\mathbf{V} = [0\ 1]$

3(b) (2 points) Your specification of the observer, to obtain an estimate $\widehat{\mathbf{q}}[n]$ of the state $\mathbf{q}[n]$ (explain your choice):

Solution:.

Since $w[n]$ and $v[n]$ are noise processes that are not accessible to us, our observer takes the form:

$$\widehat{\mathbf{q}}[n+1] = \mathbf{A}\widehat{\mathbf{q}}[n] + \mathbf{b}x[n] - \mathbf{l}(y[n] - \mathbf{c}^T\widehat{\mathbf{q}}[n])$$
$$\text{i.e.,} \quad \widehat{\mathbf{q}}[n+1] = \mathbf{A}\widehat{\mathbf{q}}[n] + \mathbf{b}x[n] - \mathbf{l}(\mathbf{c}^T\mathbf{q}[n] + v[n] - \mathbf{c}^T\widehat{\mathbf{q}}[n])$$

3(c) (2 points) With $\widetilde{\mathbf{q}}[n] = \mathbf{q}[n] - \widehat{\mathbf{q}}[n]$, explain carefully why the components $\widetilde{q}_1[n]$ and $\widetilde{q}_2[n]$ of $\widetilde{\mathbf{q}}[n]$ at time n are uncorrelated with the noise terms $w[n]$ and $v[n]$ at time n (or — equivalently, of course! — explain why the components of $\widetilde{\mathbf{q}}[n+1]$ are uncorrelated with $w[n+1]$ and $v[n+1]$):

Solution:.

The state vector $\mathbf{q}[n+1]$ depends on $w[k]$ for $k \leq n$. The estimated state vector $\widehat{\mathbf{q}}[n+1]$ depends on $v[k]$ for $k \leq n$, and on $w[j]$ for $j \leq n-1$ since $\widehat{\mathbf{q}}[n+1]$ is a function of $\mathbf{q}[n]$. Hence, $\widetilde{\mathbf{q}}[n+1] = \mathbf{q}[n+1] - \widehat{\mathbf{q}}[n+1]$ depends only on $v[k]$ for $k \leq n$ and on $w[k]$ for $k \leq n$.

Since the processes $w[n]$ and $v[n]$ are white and uncorrelated with each other, we know that each of $w[n+1]$ and $v[n+1]$ is uncorrelated with $w[k]$ and $v[k]$ for $k \leq n$. Consequently, $w[n+1]$ and $v[n+1]$ are uncorrelated with $\widetilde{\mathbf{q}}[n+1]$.

3(d) (4 points) The state estimation error in 3(c) is governed by a state-space model of the form

$$\widetilde{\mathbf{q}}[n+1] = \mathbf{B}\widetilde{\mathbf{q}}[n] + \mathbf{f}w[n] + \mathbf{g}v[n] \; .$$

Determine $\mathbf{B}$, $\mathbf{f}$ and $\mathbf{g}$ in terms of previously specified quantities.

Solution:.

$$\begin{aligned}
\tilde{\mathbf{q}}[n+1] &= \mathbf{q}[n] - \widehat{\mathbf{q}}[n] \\
&= \mathbf{A}\mathbf{q}[n] + \mathbf{h}w[n] + \mathbf{b}x[n] - (\mathbf{A}\widehat{\mathbf{q}}[n] + \mathbf{b}x[n] - \mathbf{l}(y[n] - \mathbf{c}^T\widehat{\mathbf{q}}[n])) \\
&= \mathbf{A}\mathbf{q}[n] + \mathbf{h}w[n] + \mathbf{b}x[n] - (\mathbf{A}\widehat{\mathbf{q}}[n] + \mathbf{b}x[n] - \mathbf{l}(\mathbf{c}^T\mathbf{q}[n] + v[n] - \mathbf{c}^T\widehat{\mathbf{q}}[n])) \\
&= (\mathbf{A} + \mathbf{l}\mathbf{c}^T)\tilde{\mathbf{q}}[n] + \mathbf{h}w[n] + \mathbf{l}v[n] \\
&= \mathbf{B}\tilde{\mathbf{q}}[n] + \mathbf{f}w[n] + \mathbf{g}v[n]
\end{aligned}$$

So $\mathbf{B} = \mathbf{A} + \mathbf{l}\mathbf{c}^T = \begin{bmatrix} \frac{1}{2} & \frac{3}{4} + l_1 \\ 0 & 2 + l_2 \end{bmatrix}$, $\mathbf{f} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $\mathbf{g} = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}$

3(e) (5 points) Is it possible to *arbitrarily* vary the natural frequencies of the state estimation error evolution equation in 3(d) by controlling the observer gains ℓ_1 and ℓ_2? Explicitly note how your answer here is consistent with your answer to 3(a)(iv).

Solution: The eigenvalues of the matrix $\mathbf{B}$, which governs the state estimation error dynamics, are $\frac{1}{2}$ and $2+l_2$. So, only one of the eigenvalues can be arbitrarily placed using the observer gain vector $\mathbf{l}$. This is consistent with λ_2 being the only mode observable from the output $y[n]$.

What constraints, if any, on ℓ_1 and ℓ_2 must be satisfied to make the error evolution equation asymptotically stable?

Solution: For the error evolution equation to be asymptotically stable, we need the eigenvalues of the matrix $\mathbf{B}$ to be within the unit circle. One eigenvalue is already within the unit circle and fixed at $\frac{1}{2}$. For the second eigenvalue to be within the unit circle we need $|2 + l_2| < 1$ or $-3 < l_2 < -1$. There is no constraint on the value of l_1.

Would the choice $\ell_2 = 0$ allow you to obtain a good state estimate? — explain.

Solution: No. Choosing $l_2 = 0$ would result in a exponentially growing error in estimating the state $q_2[n]$.

If you have done things correctly, you should find that choosing $\ell_1 = -\frac{3}{4}$ makes the matrix **B** in part 3(d) a diagonal matrix. **Keep ℓ_1 fixed at $-\frac{3}{4}$ for the rest of this problem**, and also assume ℓ_2 is chosen so that the error evolution equation is asymptotically stable.

3(f) (4 points) Under the given assumptions, the mean-squared estimation errors attain constant steady-state values, $E(\widetilde{q}_1^2[n]) = \sigma_{q1}^2$ and $E(\widetilde{q}_2^2[n]) = \sigma_{q2}^2$. Find explicit expressions for σ_{q1}^2 and σ_{q2}^2, expressing them as functions of ℓ_2. [**Hint**: At steady state, $E(\widetilde{q}_1^2[n+1]) = E(\widetilde{q}_1^2[n])$ and $E(\widetilde{q}_2^2[n+1]) = E(\widetilde{q}_2^2[n])$.]

Solution: With $l_1 = -\frac{3}{4}$ the error evolution equation becomes

$$\tilde{\mathbf{q}}[n+1] = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 2+l_2 \end{bmatrix} \tilde{\mathbf{q}}[n] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w[n] + \begin{bmatrix} -\frac{3}{4} \\ l_2 \end{bmatrix} v[n]$$

For the state $\tilde{q}_1[n]$ we have

$$\begin{aligned} E[\tilde{q}_1^2[n+1]] &= E[(\frac{1}{2}q_1[n] - \frac{3}{4}v[n])^2] \\ &= \frac{1}{4}E[q_1^2[n]] + \frac{9}{16}\sigma_v^2 \end{aligned}$$

In steady state we have $E(\widetilde{q}_1^2[n+1]) = E(\widetilde{q}_1^2[n])$ so

$$\begin{aligned} \frac{3}{4}E[\tilde{q}_1^2[n]] &= \frac{9}{16}\sigma_v^2 \\ \sigma_{q_1}^2 &= \frac{3}{4}\sigma_v^2 \end{aligned}$$

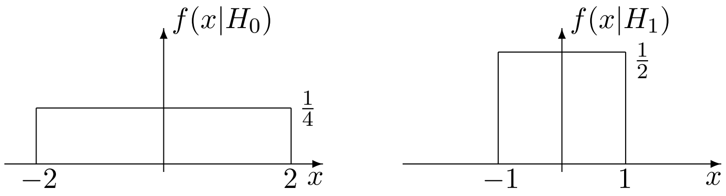
For the state $\tilde{q}_2[n]$ we have

$$\begin{aligned} E[\tilde{q}_2^2[n+1]] &= E[((2+l_2)q_2[n] + w[n] + l_2 v[n])^2] \\ &= (2+l_2)^2 E[q_2^2[n]] + \sigma_w^2 + l_2^2\sigma_v^2 \end{aligned}$$

In steady state we have $E(\widetilde{q}_2^2[n+1]) = E(\widetilde{q}_2^2[n])$ so

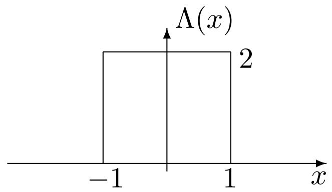
$$\sigma_{q_2}^2 = \frac{\sigma_w^2 + l_2^2\sigma_v^2}{1-(2+l_2)^2}$$

Problem 4 (20 points)



4(a) (4 points) Sketch $\Lambda(x) = \frac{f_{X|H}(x|H_1)}{f_{X|H}(x|H_0)}$ as a function of x for $-2 < x < 2$:

Solution:



4(b) (6 points)

(i) For threshold η at some value strictly above 2, determine P_D and P_{FA}:

Solution: In this case, we will always declare 'H_0' since $\Lambda(x) < \eta$. Consequently, $P_D = 0$ and $P_{FA} = 0$.

(ii) For η at some value strictly between 0 and 2, determine P_D and P_{FA}:

Solution: In this case, we will declare 'H_1' for $|x| \leq 1$ and 'H_0' for $|x| > 2$. Consequently, $P_D = 1$ and $P_{FA} = \frac{1}{2}$.

4(b) (continued)

(iii) For η at some value strictly below 0, determine P_D and P_{FA}:

Solution: In this case, we will always declare 'H_1' since $\Lambda(x) > \eta$. Consequently, $P_D = 1$ and $P_{FA} = 1$.

4(c) (2 points) If the specified limit on P_{FA} is $\beta = 0.3$, which of the choices in 4(b) can we pick, and what is the associated P_D?

Solution: Only $\eta > 2$, i.e. always declare 'H_0', results in $P_{FA} < 0.3$. However, $P_D = 0$.

4(d) (8 points) What is the probability that we get $\Lambda(X) = 0$ if H_0 holds? And what is the probability we get $\Lambda(X) = 0$ if H_1 holds?

Solution: Under the hypothesis H_0, $\Lambda(X) = 0$ when $1 < |x| < 2$. The value of x falls within this range with probability $\frac{1}{2}$, so $P(\Lambda(X) = 0 \,|\, H_0) = \frac{1}{2}$. Under the hypothesis H_1, $\Lambda(X)$ never equals 0 since $|x| \leq 1$. So, $P(\Lambda(X) = 0 \,|\, H_1) = 0$

Announce 'H_0' when $\Lambda(x) = 0$. When $\Lambda(x) > 0$, announce 'H_1' with probability α, and otherwise announce 'H_0'. What are P_D and P_{FA} with this *randomized* decision rule?

Solution:

$$P_D = \alpha P(\text{`}H_1\text{'}|H_1) = \alpha P(\Lambda > 0|H_1) = \alpha$$

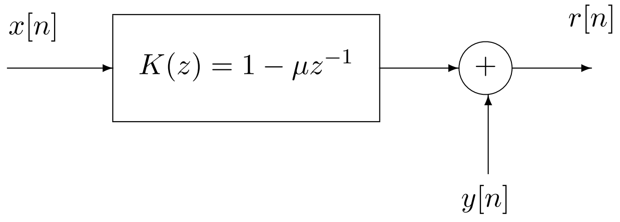
$$P_{FA} = \alpha P(\text{`}H_1\text{'}|H_0) = P(\Lambda > 0|H_0) = \frac{\alpha}{2}$$

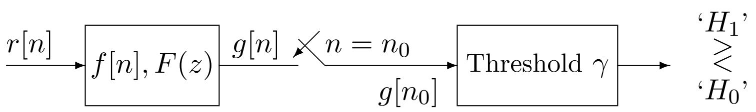
To maximize P_D while keeping $P_{FA} \leq 0.3$ with this decision rule, choose α as follows

Solution:

$P_{FA} = \frac{\alpha}{2} \leq 0.3$ so $\alpha \leq 0.6$. Since $P_D = \alpha$, we choose $\alpha = 0.6$ to maximize P_D while satisfying the constraint on P_{FA}.

Problem 5 (20 points)





5(a) (10 points) Suppose $x[n]$ is a signal that we are interested in, while $y[n]$ is a zero-mean, i.i.d., Gaussian noise process, with variance σ^2 at each instant of time.

$$\begin{aligned} H_0 : x[n] &= 0 , & P(H_0) = p_0 , \\ H_1 : x[n] &= \delta[n] , & P(H_1) = p_1 = 1 - p_0 . \end{aligned}$$

(i) Fully specify the MPE receiver when $n_0 = 0$, i.e., specify $f[n]$ or $F(z)$ and the value of γ.

Solution: The received signal $r[n]$ takes the following values under each hypothesis:

$$\begin{aligned} H_0 : r[n] &= y[n] \\ H_1 : r[n] &= \delta[n] - \mu\delta[n-1] + y[n] \end{aligned}$$

Since $y[n]$ is an i.i.d. process, we know the MPE receiver will employ a filter whose impluse response is a time-reversed replica of the pulse to be detected. So $f[n] = \delta[n] - \mu\delta[n+1]$ and $F(z) = 1 - \mu z$.

To compute the threshold γ, we need to apply the MAP detection rule. The distribution of $g[0]$ conditioned on each hypothesis is

$$f_{g[0]|H_0} = N(0, \sigma^2(1+\mu^2))$$
$$f_{g[0]|H_1} = N(1+\mu^2, \sigma^2(1+\mu^2))$$

The MAP detection rule in this case declares 'H_1' when

$$
\begin{aligned}
f_{g[0]|H_1}p_1 &> f_{g[0]|H_0}p_0 \\
\frac{e^{-\frac{1}{2\sigma^2(1+\mu^2)}(g[0]-(1+\mu^2))^2}}{e^{-\frac{1}{2\sigma^2(1+\mu^2)}(g[0])^2}} &> \frac{p_0}{p_1} \\
g[0] &> \sigma^2\ln(\frac{p_0}{p_1}) + \frac{1+\mu^2}{2} \\
g[0] &> \gamma
\end{aligned}
$$

5(a)(ii) Write down an expression for $P(`H_1\text{'}|H_0)$ and for the minimum probability of error in the case where the two hypotheses are equally likely, $p_0 = p_1$. You can write these in terms of the standard function

$$
Q(\alpha) = \frac{1}{\sqrt{2\pi}}\int_\alpha^\infty e^{-\frac{t^2}{2}}\,dt
$$

Solution: When $p_0 = p_1$, then $\gamma = \frac{1+\mu^2}{2}$. In this case $P(`H_1\text{'}|H_0)$ is

$$
\begin{aligned}
P(g[0] > \gamma|H_0) &= P(\frac{g[0]}{\sigma\sqrt{1+\mu^2}} > \frac{\frac{1+\mu^2}{2}}{\sigma\sqrt{1+\mu^2}}|H_0) \\
&= Q(\frac{\frac{1+\mu^2}{2}}{\sigma\sqrt{1+\mu^2}})
\end{aligned}
$$

The probability of error P_e is given by

$$
\begin{aligned}
P_e &= p_0P(`H_1\text{'}|H_0) + p_1P(`H_0\text{'}|H_1) \\
&= \frac{1}{2}Q(\frac{\frac{1+\mu^2}{2}}{\sigma\sqrt{1+\mu^2}}) + \frac{1}{2}(1 - Q(\frac{-\frac{1+\mu^2}{2}}{\sigma\sqrt{1+\mu^2}})) \\
&= \frac{1}{2}Q(\frac{\frac{1+\mu^2}{2}}{\sigma\sqrt{1+\mu^2}}) + \frac{1}{2}Q(\frac{\frac{1+\mu^2}{2}}{\sigma\sqrt{1+\mu^2}}) \\
&= Q(\frac{\frac{1+\mu^2}{2}}{\sigma\sqrt{1+\mu^2}})
\end{aligned}
$$

5(a)(iii) If the value of μ is changed to a new value $\overline{\mu} = \mu/2$, we can get the same probability of error as prior to the change if the noise variance changes to some new value $\overline{\sigma}^2$. Express $\overline{\sigma}$ in terms of σ:

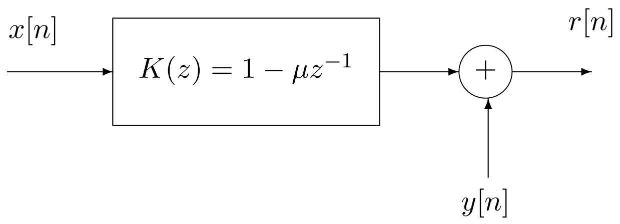
Solution: With the new parameters $\overline{\mu}$ and $\overline{\sigma}^2$, we require that the probability of error remain equal to that in part 5(a)(ii), this implies

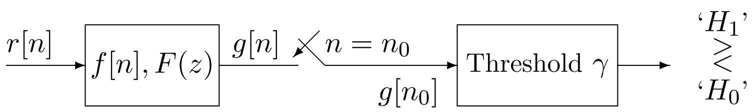
$$\begin{aligned} Q\left(\frac{\frac{1+\mu^2}{2}}{\sigma\sqrt{1+\mu^2}}\right) &= Q\left(\frac{\frac{1+\overline{\mu}^2}{2}}{\overline{\sigma}\sqrt{1+\overline{\mu}^2}}\right) \\ \frac{\frac{1+\mu^2}{2}}{\sigma\sqrt{1+\mu^2}} &= \frac{\frac{1+\overline{\mu}^2}{2}}{\overline{\sigma}\sqrt{1+\overline{\mu}^2}} \end{aligned}$$

Since $\overline{\mu} = \mu/2$

$$\begin{aligned} \frac{\frac{1+\mu^2}{2}}{\sigma\sqrt{1+\mu^2}} &= \frac{\frac{1+\frac{\mu}{2}^2}{2}}{\overline{\sigma}\sqrt{1+\frac{\mu}{2}^2}} \\ \overline{\sigma} &= \sigma\sqrt{\frac{1+(\frac{\mu}{2})^2}{1+\mu^2}} \end{aligned}$$

5(b) (10 points) Suppose now that $x[n]$ is a zero-mean, i.i.d., Gaussian noise process, with variance σ^2 at each instant of time, and that $y[n]$ is the signal we are interested in. We





have the following two hypotheses regarding $y[n]$:

$$
\begin{aligned}
H_0 : y[n] &= 0\,, & P(H_0) &= p_0\,, \\
H_1 : y[n] &= \delta[n]\,, & P(H_1) &= p_1 = 1 - p_0\,.
\end{aligned}
$$

Fully specify the MPE receiver when $n_0 = 0$, i.e., specify $f[n]$ or $F(z)$ and the value of γ for this case.

Solution: The received signal $r[n]$ takes the following values under each hypothesis:

$$
\begin{aligned}
H_0 : r[n] &= v[n] \\
H_1 : r[n] &= \delta[n] + v[n]
\end{aligned}
$$

where $v[n]$ is a colored noise process with $S_{vv}(z) = K(z)K(z^{-1})\sigma^2$. In this scenario, the MPE receiver first passes the signal $r[n]$ through a whitening filter $H_w(z) = \frac{1}{K(z)}$ in order to transform the problem into one of detecting a known pulse in a white noise process. After applying the whitening filter, the hypothesis testing problem becomes

$$
\begin{aligned}
H_0 : r[n] * h_w[n] &= v[n] * h_w[n] = x[n] \\
H_1 : r[n] * h_w[n] &= \delta[n] * h_w[n] + v[n] * h_w[n] = \mu^n u[n] + x[n]
\end{aligned}
$$

Now the receiver will employ a filter whose impluse response is a time-reversed replica of the new pulse. So $h_m[n] = \mu^{-n}u[-n]$ and $H_m(z) = \frac{1}{K(z^{-1})}$. Overall, the MPE receiver is a cascade of a whitening filter and a matched filter that is followed by a threshold test on the output at $n_0 = 0$.

$$F(z) = H_w(z)H_m(z) = \frac{1}{K(z)}\frac{1}{K(z^{-1})} = \frac{\sigma^2}{S_{vv}(z)}$$

To compute the threshold γ, we need to apply the MAP detection rule. The distribution of $g[0]$ conditioned on each hypothesis is

$$\begin{aligned} f_{g[0]|H_0} &= N(0, \frac{\sigma^2}{1-\mu^2}) \\ f_{g[0]|H_1} &= N(\frac{1}{1-\mu^2}, \frac{\sigma^2}{1-\mu^2}) \end{aligned}$$

The MAP detection rule in this case declares 'H_1' when

$$\begin{aligned} f_{g[0]|H_1}p_1 &> f_{g[0]|H_0}p_0 \\ \frac{e^{-\frac{1}{2\frac{\sigma^2}{1-\mu^2}}(g[0]-\frac{1}{1-\mu^2})^2}}{e^{-\frac{1}{2\frac{\sigma^2}{1-\mu^2}}(g[0])^2}} &> \frac{p_0}{p_1} \\ g[0] &> \sigma^2\ln(\frac{p_0}{p_1}) + \frac{1}{2(1-\mu^2)} \\ g[0] &> \gamma \end{aligned}$$

The relevant "signal energy to noise power" ratio that governs the performance of this system is

$$\frac{\frac{1}{1-\mu^2}}{\sigma^2} = \frac{1}{\sigma^2(1-\mu^2)}$$