MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.011 Introduction to Communication, Control and Signal Processing

Fall 2003

FINAL EXAM

Tuesday, December 16, 9 AM – 12 PM

- Put your name on **each** page of this booklet. Specify your recitation instructor and your recitation time.
- This is a closed book exam, but five $8\frac{1}{2}'' \times 11''$ sheets of notes (both sides) are allowed. They can be as big as $8\frac{1}{2}'' \times 11''$ or as small as you'd like and you can write on one side or two sides of each, but only five sheets are allowed.
- Everything on the notes must be in your original handwriting (i.e., material cannot be xeroxed from solutions, tables, books, etc).
- You have three hours for this exam.
- Calculators are NOT allowed.
- We will NOT provide a table of transforms.
- There are 6 problems on the exam with the percentage for each part and the total percentage for each problem as indicated. Note that the problems do not all have the same total percentage.
- Make sure you have seen all 22 numbered sides of this answer booklet.
- The problems are not in order of difficulty. We recommend that you read through all the problems, then do the problems in whatever order suits you best.
- We tried to provide ample space for you to write in. However, the space provided is not an indication of the length of the explanation required. Short, to the point, explanations are preferred to long ones that show no understanding.
- Please be neat—we can't grade what we can't decipher.

All work and answers must be in the space provided on the exam booklet. You are welcome to use scratch pages that we provide but when you hand in the exam we will not accept any pages other than the exam booklet. There will be absolutely no exceptions.

Grading on the Final Exam:
As with the other exams, in grading the final exam we will be focusing on your level of understanding of the material associated with each problem. When we grade each part of a problem we will do our best to assess, from your work, your level of understanding. On each part of an exam question we will also indicate the percentage of the total exam grade represented by that part, and your numerical score on the exam will then be calculated accordingly.

Our assessment of your level of understanding will be based upon what is given in your solution. A correct answer with no explanation will not receive full credit, and may not receive much–if any. An incorrect final answer having a solution and explanation that shows excellent understanding quite likely will receive full (or close to full) credit.

Graded Exams and Final Course Grade:
Graded exams and the final course grade can be picked up after 1 p.m. on Thursday, December 18th. If you would like your graded exam mailed to you, please leave an addressed, stamped envelope with us at the end of the exam. We'll use the envelope as is, so please be sure to address it properly and with enough postage. Also, we'll guarantee that we'll put it into the proper mailbox but won't guarantee anything beyond that. Please look over the grading of the exam before leaving. We will not consider any regrading of the exam once you take it away.

OUT OF CONSIDERATION FOR THE 6.011 STAFF, UNDER NO CIRCUMSTANCES WILL THE GRADE BE AVAILABLE BY PHONE OR EMAIL. PLEASE DON'T EVEN ASK.

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Full Name:____

	Points	Grader
1		
2(a)		
2(b)		
2(c)		
2(d)		
3(a)		
3(b)		
4(a)		
4(b)		
4(c)		
5(a)		
5(b)		
5(c)		
6(a)		
6(b)		
Total		

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No work on this page will be evaluated.

FOR THE EXAM, YOU MAY FIND SOME, NONE, OR ALL OF THE FOLLOWING USEFUL:

- Parseval's identity:

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{<2\pi>} |X(e^{j\Omega})|^2 d\Omega.$$

- Univariate Gaussian PDF: $f_X(x) = \dfrac{1}{\sqrt{2\pi}\sigma} \exp(-\frac{1}{2}(\frac{x-m}{\sigma})^2)$

- Two random variables X and Y are said to have a bivariate Gaussian joint PDF if the joint density of the *centered* (i.e. zero-mean) and *normalized* (i.e. unit-variance) random variables

$$V = \frac{X - \mu_X}{\sigma_X}$$

$$W = \frac{Y - \mu_Y}{\sigma_Y}$$

is given by

$$f_{V,W}(v,w) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp(-\frac{(v^2 - 2\rho vw + w^2)}{2(1-\rho^2)})$$

Here μ_X and μ_Y are the means of X and Y respectively, and σ_X, σ_Y are the respective standard deviations of X and Y. The number ρ is called the correlation coefficient of X and Y, and is defined by

$$\rho = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} \text{ with } \sigma_{XY} = E[XY] - \mu_X \mu_Y$$

where σ_{XY} is the covariance of X and Y.

- The Cauchy-Schwarz inequality, tells us that the following inequality holds for any two square-integrable functions $a(t)$ and $b(t)$:

$$\left(\int_{-\infty}^{\infty} a(t)b(t)dt \right)^2 \leq \int_{-\infty}^{\infty} a^2(t)dt \int_{-\infty}^{\infty} b^2(t)dt$$

with equality if and only if $a(t) = kb(t)$ where k is a constant.

- MMSE for two bivariate Gaussian random variables, X and Y:

$$\widehat{Y}_{MMSE}(x) = \mu_Y + \rho \frac{\sigma_Y}{\sigma_X}(x - \mu_X)$$

Problem 1 (10%)

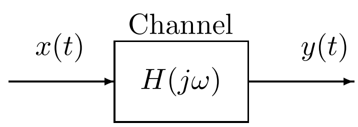
Suppose we apply the modulated signal

$$x(t) = m(t)\,\cos(\omega_0 t)$$

to the input of an LTI communication channel with frequency response $H(j\omega)$, where the modulating signal is

$$m(t) = \frac{\sin(\pi t/T)}{\pi t/T}.$$

Assume $(1/T) = 75$ kHz and $(\omega_0/2\pi) = 1300$ kHz.



Find an *approximate* (but reasonably accurate) time-domain expression for the output $y(t)$ of the channel if the channel characteristics are as shown in Figure 1-1. Also state what features of the signal and/or channel make your approximation reasonable.

$y(t) =$

What features make the approximation reasonable?

Explanation: (Specifically, indicate on Figure 1-1 how you arrived at your conclusions.)

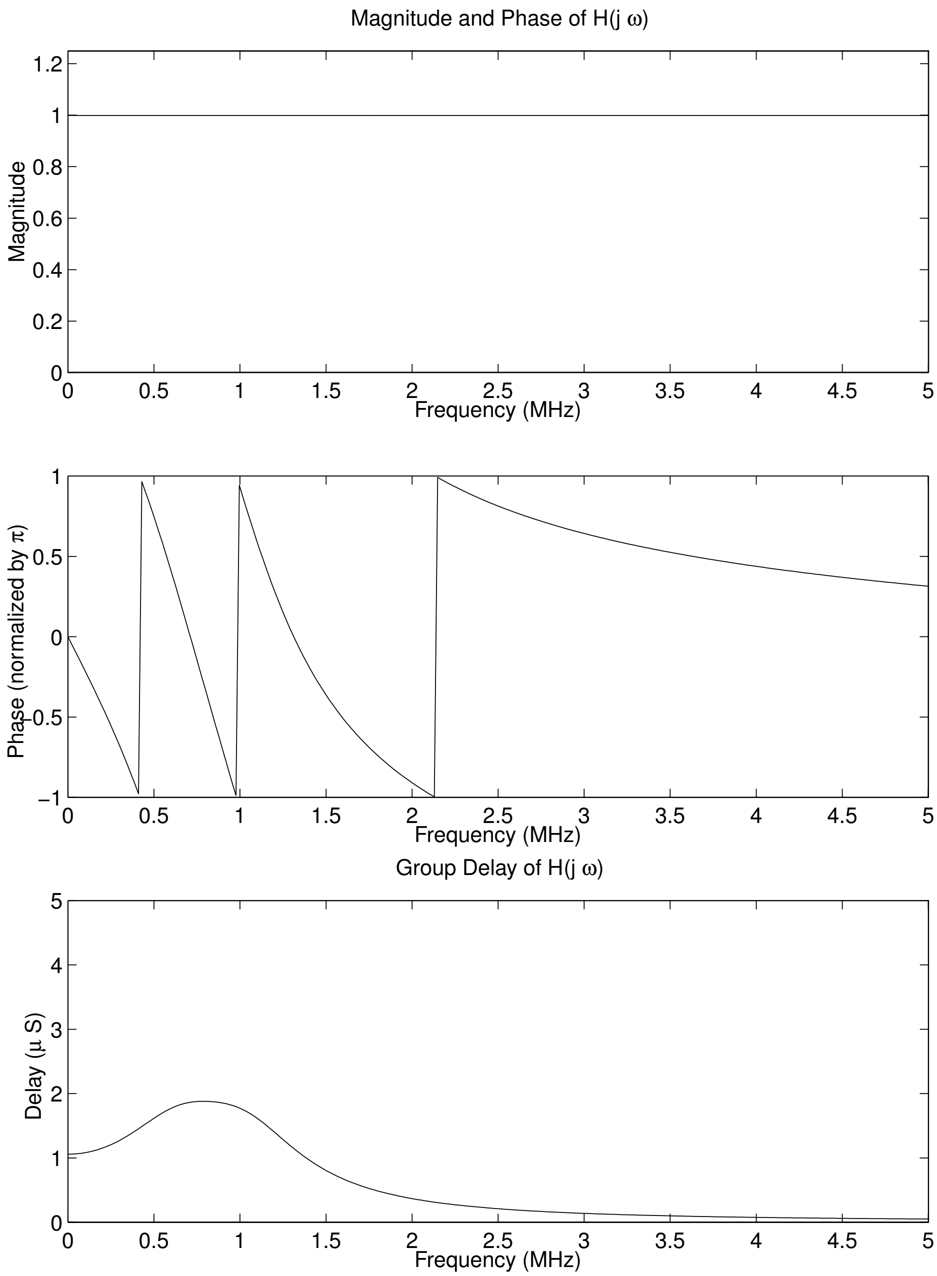


Figure 1-1: Channel characteristics

Problem 2 (20%)

The state evolution of a discrete-time system is described by the state evolution equation

$$\mathbf{q}[n+1] \quad = \quad \mathbf{A}\mathbf{q}[n] + \mathbf{b}x[n]$$

with

$$\mathbf{A} = \begin{bmatrix} 1 & \beta \\ -\alpha & 1-\alpha\beta \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

where α and β are real-valued constants.

The matrix A has the eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 0.5$. The eigenvector corresponding to eigenvalue λ_1 is $\mathbf{v_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

(a) (5%) Determine the parameters α and β and the eigenvector $\mathbf{v_2}$.

$\alpha =$ $\qquad\qquad$ $\beta =$ $\qquad\qquad$ $\mathbf{v_2} =$

Work to be looked at:

(b) (5%) Is the system asymptotically stable? (i.e. with zero input will the state vector asymptotically decay to zero for every choice of initial conditions?) Circle your answer:

YES $\qquad\qquad$ NO

Work to be looked at:

The state of the system is measured at each time n and used for state feedback to the input according to the relation

$$x[n] = \mathbf{g}^T \mathbf{q}[n] + p[n]$$

where p[n] is the input to the closed-loop system.

(c) (5%) If possible, determine a $\mathbf{g}^T$, so that the eigenvalues of the closed-loop state feedback system are at 0.5 and 0. If it is not possible, clearly explain why not.

Possible? YES NO

If yes, $g_1 =$ $g_2 =$

If no, explain why not:

Work to be looked at:

(d) (5%) With an appropriate choice of g_1 and g_2 can the eigenvalues of the closed loop state feedback system be arbitrarily placed? Please circle your answer:

YES NO

Work to be looked at:

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Problem 3 (16%)

Figure 3-1 shows the channel model for a wireless communication system with a direct path and a reflected path.

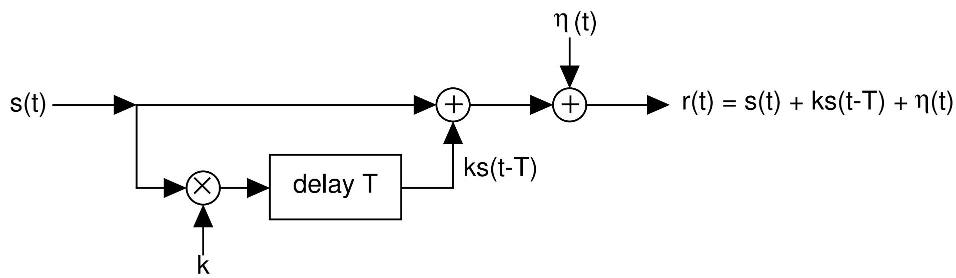

Figure 3-1: Channel model for a wireless communication system.

In this figure:

- The channel input $s(t)$ is a zero-mean, wide-sense stationary random process whose power spectral density is
$$S_{ss}(j\omega) = \frac{2\lambda\sigma_s^2}{\omega^2 + \lambda^2},$$
where λ and σ_s^2 are positive constants.
- The channel output is $r(t) = s(t) + ks(t - T) + \eta(t)$, where k is a positive constant representing the strength of the reflected path, T is a positive constant representing the delay of the reflected path relative to the direct path, and $\eta(t)$ is the channel noise.
- The channel noise $\eta(t)$ is zero-mean, wide-sense stationary white noise that is statistically independent of the process $s(t)$. The power spectral density of $\eta(t)$ is $S_{\eta\eta}(j\omega) = N$.

(a) (8%) Determine $S_{rr}(j\omega)$, the power spectral density of $r(t)$.

$S_{rr}(j\omega) =$

Work to be looked at:

(b) (8%) We want to pass $r(t)$ through a linear time-invariant filter with frequency response $H(j\omega)$ to obtain an estimate of the channel input $s(t)$. Determine the frequency response $H(j\omega)$ that minimizes the mean squared-error of this estimate.

$H(j\omega) =$

Work to be looked at:

Problem 4 (18%)

Consider a binary hypothesis testing problem in which a receiver observes a random variable R. Based on this observation the receiver decides which one of two hypotheses — denoted by H_0 and H_1 — to declare as true. The receiver can be tuned to operate on any point on the Receiver Operating Characteristic (ROC). The ROC for this receiver is given by $P_D = \sqrt{P_{FA}}$, where $P_D = P(\text{'}H_1\text{'}|H_1)$ and $P_{FA} = P(\text{'}H_1\text{'}|H_0)$. The probability of error P_e of the receiver is defined as the probability of declaring 'H_0' and H_1 is true or declaring 'H_1' and H_0 is true.

(a) (6%) For this part, suppose that the prior probability of hypothesis H_0 being true is $P(H_0) = 3/4$ and that the receiver is tuned to operate at the point $P_D = 1/2$ on the ROC curve. Determine P_{FA} and P_e at that operating point.

$P_{FA} =$ $P_e =$

Work to be looked at:

(b) (6%) For the prior probability of H_0 given in (a) (i.e. $P(H_0) = 3/4$), there is an operating point on the ROC curve that minimizes the overall probability of error P_e. Determine P_D if the receiver operates at that point.

$P_D =$

Work to be looked at:

(c) (6%) Now let $P(H_0) = 1/4$. Determine P_D and P_{FA} on the ROC curve and the corresponding P_e such that P_e is minimized.

$P_D =$ $P_{FA} =$ $P_e =$

Work to be looked at:

Problem 5 (20%)

In Figure 5-1 we show a PAM system in which the transmitted sequence $a[n]$ is deterministic (i.e. not a random process) and is continuous in amplitude.

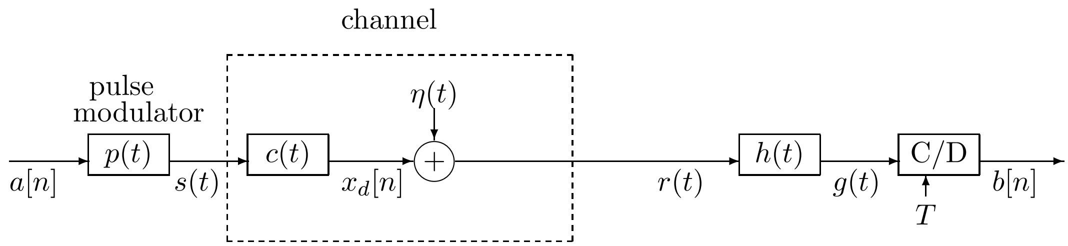

Figure 5-1: PAM System

The channel is modeled as an LTI system with impulse response $c(t)$ and additive noise $\eta(t)$. The received signal $r(t)$ is processed with a LTI filter $h(t)$ and then sampled to obtain $b[n]$.

The associated relationships are:

- $s(t) = \sum_{n=-\infty}^{\infty} a[n]p(t - nT)$
- $p_c(t)$ is defined as $p(t) * c(t)$
- $r(t) = \sum_{n=-\infty}^{\infty} a[n]p_c(t - nT) + \eta(t)$
- $g(t) = h(t) * r(t)$
- $b[n] = g(nT)$
- $\eta(t)$ is zero-mean wide-sense stationary with autocorrelation function $R_{\eta\eta}(\tau) = N\delta(\tau)$.

(a) (6%) **For this part only**

$\eta(t) = 0$ (i.e. $N = 0$)

$C(j\omega) = e^{-j\omega/2}$

$h(t) = \delta(t)$

$p(t)$ is as shown in figure 5-2.

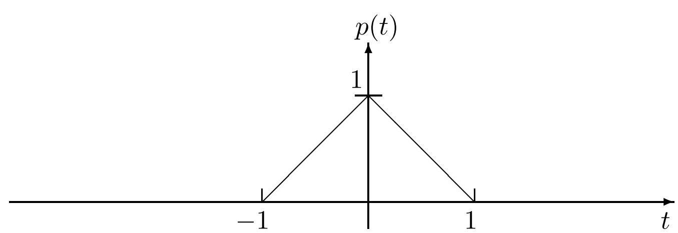

Figure 5-2: $p(t)$.

Determine the fastest symbol rate ($1/T$) so that there is no intersymbol interference in $g(t)$ i.e. so that $b[n] = ka[n]$ where k is a constant. Also, determine the value of k.

Fastest symbol rate= $k =$

Work to be looked at:

For the remainder of this problem assume that there is no intersymbol interference in $r(t)$ or $g(t)$.

(b) (8%) Determine the mean and variance of $b[0]$ in terms of $p_c(t), h(t), N$ and $a[n]$.

$E\{b[0]\} =$ $\text{var}\{b[0]\} =$

Work to be looked at:

(c) (6%) Determine $h(t)$ in terms of $p_c(t)$ so that $E\{b[0]\} = a[0]$ and the variance of $b[0]$ is minimized.

$h(t) =$

Work to be looked at:

Problem 6 (16%)

Figure 6-1 shows a baseband model for a communication system that employs in-line amplification. For simplicity, only transmission of a single bit is considered.

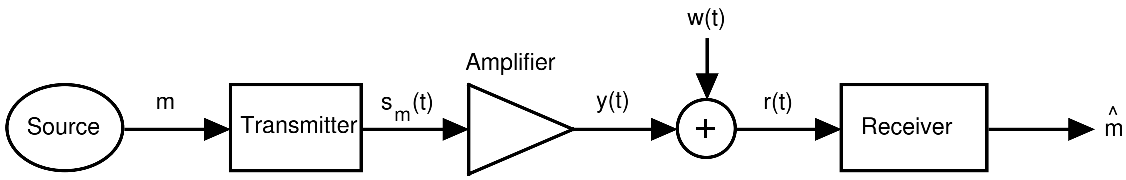

Figure 6-1: Baseband model for a communication system with in-line amplification.

In this figure:

- m is a binary symbol that is equally likely to be 0 or 1.
- $s_m(t)$ is the transmitter waveform that is used to convey the message m. The waveforms $s_0(t)$ and $s_1(t)$ associated with messages $m = 0$ and $m = 1$, respectively, are shown in Figure 6-2.

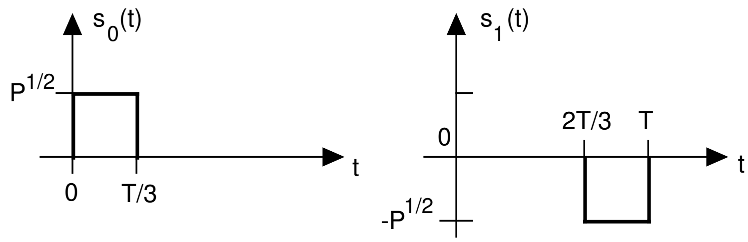

Figure 6-2: Transmitter waveforms $s_0(t)$ and $s_1(t)$; P is the transmitter's peak instantaneous power.

- The amplifier output $y(t)$ is given by

$$y(t) = \sqrt{G}s_m(t) + v(t),$$

  where $G > 1$ is the power gain of the amplifier and $v(t)$ is noise that the amplifier injects. $v(t)$ is a zero-mean, white Gaussian noise process that is statistically independent of the message m and has power spectral density $S_{vv}(j\omega) = N_v$.

- $w(t)$ is a zero-mean, white Gaussian noise process that is generated by the receiver electronics. It is statistically independent of the message m and the amplifier's noise process $v(t)$. The power spectral density of $w(t)$ is $S_{ww}(j\omega) = N_w$.
- The receiver, shown in detail in Figure 6-3, filters and then samples the received waveform, $r(t) = y(t) + w(t)$, to obtain the random variable $Z = z(T)$. The receiver's output is its decision, $\hat{m} = 0$ or 1, as to which message was sent, based on the value of Z.

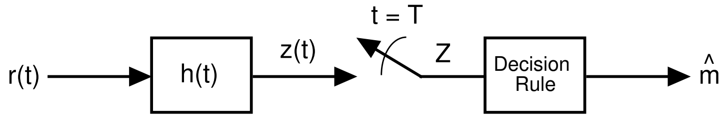

Figure 6-3: Structure of the receiver block from Figure 1.

- The receiver filter has impulse response

$$h(t) = \begin{cases} 1/\sqrt{T}, & \text{for } 0 \le t \le T, \\ 0, & \text{otherwise,} \end{cases}$$

as shown in Figure 4.

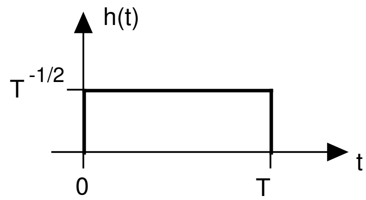

Figure 6-4: Impulse response $h(t)$ of the receiver filter.

(a) (8%) Determine the conditional probability densities for Z given $m = 0$ and $m = 1$, i.e., determine $f_{Z|m=0}(\, z \mid m = 0\,)$ and $f_{Z|m=1}(\, z \mid m = 1\,)$.

$f_{Z|m=0}(\, z \mid m = 0\,) =$

$f_{Z|m=1}(\, z \mid m = 1\,) =$

Work to be looked at:

Work to be looked at (continued):

(b) (8%) Determine the minimum error probability decision rule for deciding $\hat{m} = 0$ or $\hat{m} = 1$. Reduce your rule to a threshold test on Z.

Decision rule:

Work to be looked at:

THE END