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6.00 Notes On Big-O Notation

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See also http://en.wikipedia.org/wiki/Big_O_notation

- We use big-O notation in the analysis of algorithms to describe an algorithm's usage of computational resources, in a way that is independent of computer architecture or clock rate.
- The worst case running time, or memory usage, of an algorithm is often expressed as a function of the length of its input using big O notation.
  - In 6.00 we generally seek to analyze the worst-case running time. However it is not unusual to see a big-O analysis of memory usage.
  - An expression in big-O notation is expressed as a capital letter "O", followed by a function (generally) in terms of the variable n, which is understood to be the size of the input to the function you are analyzing.
  - This looks like: $O(n)$.
  - If we see a statement such as: $f(x)$ is $O(n)$ it can be read as "f of x is big Oh of n"; it is understood, then, that the number of steps to run $f(x)$ is linear with respect to $|x|$, the size of the input x.
- A description of a function in terms of big O notation only provides an *upper bound* on the growth rate of the function.
  - This means that a function that is $O(n)$ is also, technically, $O(n^2)$, $O(n^3)$, etc
  - However, we generally seek to provide the tightest possible bound. If you say an algorithm is $O(n^3)$, but it is also $O(n^2)$, it is generally best to say $O(n^2)$.
- Why do we use big-O notation? big-O notation allows us to compare different approaches for solving problems, and predict how long it might take to run an algorithm on a very large input.

  With big-O notation we are particularly concerned with the *scalability* of our functions. big-O bounds may not reveal the fastest algorithm for small inputs (for example, remember that for $x < 0.5$, $x^3 < x^2$) but will accurately predict the long-term behavior of the algorithm.
  - This is particularly important in the realm of scientific computing: for example, doing analysis on the human genome or data from Hubble involves input (arrays or lists) of size well into the tens of millions (of base pairs, pixels, etc).

- At this scale it becomes easy to see why big O notation is helpful. Say you're running a program to analyze base pairs and have two different implementations: one is $O(n \lg n)$ and the other is $O(n^3)$. Even without knowing how fast of a computer you're using, it's easy to see that the first algorithm will be $n^3/(n \lg n) = n^2/\lg n$ faster than the second, which is a BIG difference at input that size.

big-O notation is widespread wherever we talk about algorithms. If you take any Course 6 classes in the future, or do anything involving algorithms in the future, you will run into big-O notation again.

- Some common bounds you may see, in order from smallest to largest:
  - $O(1)$: Constant time. $O(1) = O(10) = O(2^{100})$ - why? Even though the constants are huge, they are still *constant*. Thus if you have an algorithm that takes 2^{100} discreet steps, regardless of the size of the input, the algorithm is still $O(1)$ - it runs in constant time; it is *not dependent upon the size of the input.*
  - $O(\lg n)$: Logarithmic time. This is faster than linear time; $O(\log_{10} n) = O(\ln n) = O(\lg n)$ (traditionally in Computer Science we are most concerned with $\lg n$, which is the base-2 logarithm – why is this the case?). The fastest time bound for search.
  - $O(n)$: Linear time. Usually something when you need to examine every single bit of your input.
  - $O(n \lg n)$: This is the fastest time bound we can currently achieve for sorting a list of elements.
  - $O(n^2)$: Quadratic time. Often this is the bound when we have nested loops.
  - $O(2^n)$: Really, REALLY big! A number raised to the power of n is slower than n raised to any power.

- Some questions for you:
  1. Does $O(100n^2) = O(n^2)$?
  2. Does $O(\frac{1}{4}n^3) = O(n^3)$?
  3. Does $O(n) + O(n) = O(n)$?

The answers to all of these are Yes! Why? big-O notation is concerned with the long-term, or *limiting*, behavior of functions. If you're familiar with limits, this will make sense - recall that

$$\lim_{x\to\infty} x^2 = \lim_{x\to\infty} 100x^2 = \infty$$

basically, go out far enough and we can't see a distinction between $100x^2$ and x^2. So, when we talk about big-O notation, we always *drop coefficient multipliers* - because they don't make a difference. Thus, if you're analysing your function and you get that it is $O(n) + O(n)$, that doesn't equal $O(2n)$ - we simply say it is $O(n)$.

One more question for you: Does $O(100n^2 + \frac{1}{4}n^3) = O(n^3)$?

Again, the answer to this is Yes! Because we are only concerned with how our algorithm behaves for very large values of n, when n is big enough, the n^3 term will always dominate the n^2 term, regardless of the coefficient on either of them.

In general, you will always say a function is big-O of its largest factor - for example, if something is $O(n^2 + n \lg n + 100)$ we say it is $O(n^2)$. Constant terms, no matter how huge, are always dropped if a variable term is present - so $O(800 \lg n + 73891) = O(\lg n)$, while $O(73891)$ by itself, with no variable terms present, is $O(1)$.

See the graphs generated by the file `bigO_plots.py` for a more visual explanation of the limiting behavior we're talking about here. Figures 1, 2, and 3 illustrate why we drop coefficients, while figure 4 illustrates how the biggest term will dominate smaller ones.

Now you should understand the What and the Why of big-O notation, as well as How we describe something in big-O terms. But How do we get the bounds in the first place?? Let's go through some examples.

1. We consider all mathematical operations to be constant time ($O(1)$) operations. So the following functions are all considered to be $O(1)$ in complexity:

```
def inc(x):
   return x+1

def mul(x, y):
   return x*y

def foo(x):
   y = x*77.3
   return x/8.2

def bar(x, y):
   z = x + y
   w = x * y
   q = (w**z) % 870
   return 9*q
```

2. Functions containing for loops that go through the whole input are generally $O(n)$. For example, above we defined a function `mul` that was constant-time as it used the built-in Python operator `*`. If we define our own multiplication function that doesn't use `*`, it will not be $O(1)$ anymore:

```
def mul2(x, y):
   result = 0
   for i in range(y):
      result += x
   return result
```

   Here, this function is $O(y)$ - the way we've defined it is dependent on the size of the input y, because we execute the `for` loop y times, and each time through the `for` loop we execute a constant-time operation.

3. Consider the following code:

```
def factorial(n):
   result = 1
   for num in range(1, n+1):
      result *= num
   return num
```

   What is the big-O bound on `factorial`?

4. Consider the following code:

```
def factorial2(n):
   result = 1
   count = 0
   for num in range(1, n+1):
      result *= num
      count += 1
   return num
```

   What is the big-O bound on `factorial2`?

5. The complexity of conditionals depends on what the condition is. The complexity of the condition can be constant, linear, or even worse - it all depends on what the condition is.

```
def count_ts(a_str):
   count = 0
   for char in a_str:
      if char == 't':
         count += 1
   return count
```

In this example, we used an `if` statement. The analysis of the runtime of a conditional is highly dependent upon what the conditional's condition actually is; checking if one character is equal to another is a constant-time operation, so this example is linear with respect to the size of `a_str`. So, if we let $n = |\texttt{a\_str}|$, this function is $O(n)$.

Now consider this code:

```
def count_same_ltrs(a_str, b_str):
   count = 0
   for char in a_str:
      if char in b_str:
         count += 1
   return count
```

This code looks very similar to the function `count_ts`, but it is actually very different! The conditional checks if `char in b_str` - this check requires us, in the *worst case*, to check every single character in `b_str`! Why do we care about the worst case? Because big-O notation is an upper bound on the *worst-case running time*. Sometimes analysis becomes easier if you ask yourself, what input could I give this to achieve the maximum number of steps? For the conditional, the worst-case occurs when `char` is **not** in `b_str` - then we have to look at every letter in `b_str` before we can return False.

So, what is the complexity of this function? Let $n = |\texttt{a\_str}|$ and $m = |\texttt{b\_str}|$. Then, the `for` loop is $O(n)$. Each iteration of the `for` loop executes a conditional check that is, in the worst case, $O(m)$. Since we execute an $O(m)$ check $O(n)$ time, we say this function is $O(nm)$.

6. While loops: With while loops you have to combine the analysis of a conditional with one of a for loop.

   ```
   def factorial3(n):
      result = 1
      while n > 0:
         result *= n
         n -= 1
      return result
   ```

   What is the complexity of `factorial3`?

   ```
   def char_split(a_str):
      result = []
      index = 0
      while len(a_str) != len(result):
         result.append(a_str[index])
         index += 1
      return result
   ```

   In Python, `len` is a constant-time operation. So is string indexing (this is because strings are immutable) and list appending. So, what is the time complexity of `char_split`?

   If you are curious, there is a little more information on Python operator complexity here:
   http://wiki.python.org/moin/TimeComplexity - some notes: (1) CPython just means "Python written in the C language". You are actually using CPython. (2) If you are asked to find the worst-case complexity, you want to use the Worst Case bounds. (3) Note that operations such as slicing and copying aren't $O(1)$ operations.

7. Nested for loops - anytime you're dealing with nested loops, work from the inside out. Figure out the complexity of the innermost loop, then go out a level and multiply (this is similar to the second piece of code in Example 5). So, what is the time complexity of this code fragment, if we let $n = |z|$?

   ```
   result = 0
   for i in range(z):
      for j in range(z):
         result += (i*j)
   ```

8. Recursion. Recursion can be tricky to figure out; think of recursion like a tree. If the tree has lots of branches, it will be more complex than one that has very few branches.

   Consider recursive factorial:

```
def r_factorial(n):
   if n <= 0:
      return 1
   else:
      return n*r_factorial(n-1)
```

   What is the time complexity of this? The time complexity of `r_factorial` will be dependent upon the number of times it is called. If we look at the recursive call, we notice that it is: `r_factorial(n-1)`. This means that, every time we call `r_factorial`, we make a recursive call to a subproblem of size $n-1$. So given an input of size n, we make the recursive call to subproblem of size $n-1$, which makes a call to subproblem of size $n-2$, which makes a call to subproblem of size $n-3$, $\ldots$ see a pattern? We'll have to do this until we make a call to $n-n=0$ before we hit the base case - or, n calls. So, `r_factorial` is $O(n)$. There is a direct correlation from this recursive call to the iterative loop `for i in range(n, 0, -1)`.

   In general, we can say that any recursive function $g(x)$ whose recursive call is on a subproblem of size $x-1$ will have a linear time bound, assuming that the rest of the recursive call is $O(1)$ in complexity (this was the case here, because the `n*` factor was $O(1)$).

   How about this function?

```
def foo(n):
   if n <= 1:
      return 1
   return foo(n/2) + 1
```

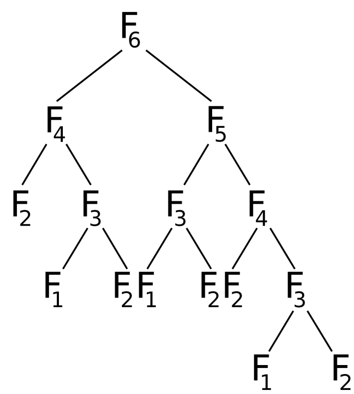
   In this problem, the recursive call is to a subproblem of size $n/2$. How can we visualize this? First we make a call to a problem of size n, which calls a subproblem of size $n/2$, which calls a subproblem of size $n/4$, which calls a subproblem of size $n/(2^3)$, $\ldots$ See the pattern yet? We can make the intuition that we'll need to make recursive calls until $n=1$, which will happen when $n/2^x=1$.

   So, to figure out how many steps this takes, simply solve for x in terms of n:

$$\frac{n}{2^x} = 1$$
$$n = 2^x$$
$$\log_2 n = \log_2(2^x)$$
$$\therefore x = \log_2 n$$

So, it'll take $\log_2 n$ steps to solve this recursive equation. In general, we can say that if a recursive function $g(x)$ makes a recursive call to a subproblem of size x/b, the complexity of the function will be $\log_b n$. Again, this is assuming that the remainder of the recursive function has complexity of $O(1)$.

Finally, how do we deal with the complexity of something like Fibonacci? The recursive call to Fibonacci is $\mathtt{fib}(n) = \mathtt{fib}(n-1) + \mathtt{fib}(n-2)$. This may initially seem linear, but it's not. If you draw this in a tree fashion, you get something like:



The *depth* of this tree (the number of levels it has) is n, and at each level we see a branching factor of two (every call to fib generates two more calls to fib). Thus, a loose bound on fib is $O(2^n)$. In fact, there exists a tighter bound on Fibonacci involving the Golden Ratio; Google for "Fibonacci complexity" to find out more if you're interested in maths : D