14.75: Sometimes it Gets Complicated

Condorcet's Paradox and Arrow's Impossibility Theorem

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Condorcet's Paradox

- Let's go back to our 3 option example we talked about a few lectures ago. There are 3 options, A, B, and C.
- In principle, you could imagine 6 different ways preferences could be ordered:
  - $A > B > C$
  - $A > C > B$
  - $B > A > C$
  - $B > C > A$
  - $C > A > B$
  - $C > B > A$

Condorcet's Paradox

- Suppose that we have 3 people with preferences as follows:
  1. $A > B > C$
  2. $B > C > A$
  3. $C > A > B$
- Suppose we had a vote between A and B. Who would win? A
- Suppose we had a vote between B and C. Who would win? B
- Suppose we had a vote between C and A. Who would win? C
- Who would you expect to win if there was transitivity? A.
- But we do not have transitivity. Instead we have what's called a *Condorcet Cycle*

$$A \succ B \succ C \succ A$$

  where I'm using $\succ$ to denote the social preference ordering by majority votes.

Condorcet Cycles and Condorcet Winners

- What we just saw is an example of Condorcet's Paradox
- To be precise here are some definitions

Definition (Condorcet Winner)

A *Condorcet Winner* is an alternative such that it gains a majority of votes when paired against each of the other alternatives.

Definition (Condorcet Cycles)

A *Condorcet Cycle* occurs when there is a violation of transitivity in the social preference ordering.

Example (Condorcet Cycle)

With three alternatives $x \succ y \succ z \succ x$.
With four alternatives, you could have $x \succ y \succ z \succ q \succ x$.

Condorcet's Paradox and Condorcet Winners

Theorem

If a Condorcet Cycle occurs, then there is no Condorcet Winner.

Proof.

Consider a Condorcet Cycle of arbitrary length of the form

$$x \succ a_j \succ ... \succ a_k \succ x$$

Is x a Condorcet Winner?
No, because $x \succ a_j$ and $a_k \succ x$.
Is anything else a Condorcet Winner?
No, because there is alternative that defeats it from the assumption that $x \succ a_j \succ ... \succ a_k \succ x$. □

Condorcet's Paradox and Condorcet Winners

Example

Consider the case with 3 alternatives.
Suppose that $x \succ y \succ z \succ x$.
Is x a Condorcet Winner?
No, because $z \succ x$.
Is anybody else a Condorcet Winner?
No, because $x \succ y$ (so y is not) and $y \succ z$ (so z is not).

Condorcet's Paradox and Condorcet Winners

Theorem

A Condorcet Cycle occurs whenever there is not a Condorcet Winner.

- I'll do the cases where there are 3 and 4 alternatives, although I believe the theorem is general.

Proof.

[Proof (3 alternatives)] Suppose there is not a Condorcet Winner.
Then we know there are at least 3 alternatives such that $x \succ y$ and $z \succ x$.
Why? Otherwise, x would be a Condorcet Winner.
What about y vs z? We know that $y \succ z$ because if $z \succ y$ then z would be a Condorcet Winner.
But now we have that $x \succ y$,$z \succ x$, and $y \succ z$. So that implies that $x \succ y \succ z \succ x$ and we have a Condorcet Cycle. □

Condorcet's Paradox and Condorcet Winners

Proof.

[Proof (4 alternatives)] Supper there is a not a Condorcet Winner. Then we know there are at least 3 alternatives such that $x \succ y$ and $z \succ x$.
Suppose that $y \succ z$. Then $x \succ y \succ z \succ x$ and we're done.
So suppose that $z \succ y$. So now we have $x \succ y$, $z \succ x$, and $z \succ y$.
Clearly we need to have $q \succ z$ or else z would be a Condorcet winner.
So now we have $x \succ y, z \succ x, z \succ y, q \succ z$. So now we have
$q \succ z \succ x \succ y$.
But something (either x or y) must be preferred to q or else q would be a Condorcet winner.
If it's x, then we have $q \succ z \succ x \succ q$ and we're done.
If it's y, then we have $q \succ z \succ x \succ y \succ q$ and we're done. $\square$

Agenda setting and Condorcet Cycles

- If there are Condorcet Cycles, then the order in which we vote makes a huge difference.
- Recall our example of preferences that generate a Condorcet Cycle
  1. $A > B > C$
  2. $B > C > A$
  3. $C > A > B$

  Suppose we first voted on A vs. B, and then voted on the winner vs. C. What would happen?
  - $A \succ B$. Then $C \succ A$. So we'd end up with C.
- Suppose instead we voted on A vs. C, and then voted on the winner vs. B. What would happen?
  - $C \succ A$. Then $B \succ C$. So we'd end up with B.
- Suppose instead we voted on B vs. C, and then voted on the winner vs. A. What would happen?
  - $B \succ C$. Then $A \succ C$. So we'd end up with A.

Agenda setting and Condorcet Cycles

- This example showed that the order in which we schedule votes can make a huge difference in outcomes
  - In fact, in this particular case, the person who decides which order we vote on things can end up with any outcome they want!

- This example illustrates the power of agenda setting – i.e., deciding which options we consider and in what order we consider them
- Who are agenda setters in real life?
  - In Congress, the Speaker of the House and the Senate Majority Leader set the agenda. In fact, that's there main power.
  - The President also has agenda setting powers (in a more informal sense)
  - You can't accomplish everything as agenda setter – but you can see how it's very powerful

What's the right thing to do when you have Condorcet Cycles?

- Given that Condorcet Cycles can exist, so there is no clear Condorcet winner, what is the "right" way for societies to make decisions?
    - I.e. is there some way to aggregate preferences that is "best" in some sense?
- The answer is, there is no right answer.
- Nothing is perfect.
- This is one of the most famous results in social choice theory, and it's called Arrow's Impossibility Theorem.

Arrow's Impossibility Theorem

- To state the theorem we need a few definitions:
- Denote the set of alternatives by A, the members of society as G, and the social decision rule as $\succ$.
- An individual's preferences are "*rational*" if they are complete and transitive. That is,
  - For any two alternatives, a and b, each individual in society can (weakly) rank them, i.e. $a > b$, $a = b$, or $b > a$.
  - And, for any three alternatives, a, b, c, if $a > b$ and $b > c$, then $a > c$. (and likewise for weak inequalities)

Arrow's Impossibility Theorem

- To state the theorem we need a few definitions:
  - Assumption 1 (Universal Domain). We assume that all individuals i have rational preferences over all the alternatives in A, but beyond that, they can have any set of rational orderings.
  - Assumption 2 (Pareto Optimaility). If every member of G prefers a to b (i.e. if $a >_i b \; \forall i$), then the social decision rule must prefer a to be ($a \succ b$).
  - Assumption 3 (Independence of Irrelevant Alternatives). Suppose we have two different societies G and G', but within G and G', everyone has the same orderings of alternatives a and b. Then if $a \succ_G b$, then $a \succ_{G'} b$.
    - That is, if everyone in G and G' have the same orderings of alternatives a and b, the social ordering between a and b must be the same, even if members of G and G' have different rankings of other alternatives c.
  - Assumption 4 (No dictatorship). There is no particular individual $i^* \in G$ such that the preferences of i^* determine the social ranking $\succ$, regardless of other group members. That is, nobody is the dictator

Arrow's Impossibility Theorem

- We can now state the theorem:

Theorem (Arrow's Impossibility Theorem)

There is no social ranking function $\succ$ such that for any group G whose members all have rational preferences, $\succ$ is a rational (transitive) ranking and satisfies the Universal Domain, Pareto Optimality, Independence of Irrelevant Alternatives, and No Dictatorship assumptions.

- What does this mean?
- It means that the problem of Condorcet Cycles and agenda setting is a very deep, fundamental problem.
- If you want a social ordering that has Universal Domain, Pareto Optimality, Independence of Irrelevant Alternatives, and No Dictatorship, you can't also have transitivity – you will get cycles.

Single-peakedness

- Recall our example of preferences that generate a Condorcet Cycle
  1. $A > B > C$
  2. $B > C > A$
  3. $C > A > B$
- Are these preferences single-peaked?

Single-peakedness and Condorcet Winners

Theorem

If preferences are single-peaked, then the ideal point of the median voter is (weakly) a Condorcet Winner.

Proof.

Denote by b_{median} the ideal point of the median voter. Consider any alternative $a < b_{median}$.
All voters with ideal points $b_j > b_{median}$ will prefer b_{median} to a. Since this is at least 50% of the voters (by the definition of median), we know that $b_{median} \succeq a$.
Same is true for all alternatives $a > b_{median}$.
Thus there is no alternative that can strictly defeat b_{median}. □

- This is why single-peaked preferences are so useful – they mean that we don't have to worry about Condorcet Cycles, since they guarantee a Condorcet Winner

Are Condorcet Cycles a problem in practice?

- As we saw, sometimes we have Condorcet Cycles and sometimes we don't:
- Has Condorcet Cycles:
    1. $A > B > C$
    2. $B > C > A$
    3. $C > A > B$
- Doesn't have Condorcet Cycles:
    1. $A > B > C$
    2. $B > C > A$
    3. $C > B > A$

Are Condorcet Cycles a problem in practice?

- We can ask the question, if we have i individuals and j alternatives, what fraction of possible rankings exhibit Condorcet Cycles
- For example, with 3 individuals and 3 alternatives, there are $3 * 6 = 18$ different configurations (non-unique, since some of these are duplicates)
- Of them only 1 has a Condorcet Cycle (the one I picked)

Are Condorcet Cycles a problem in practice?

- More generally, this table shows what fraction of the time Condorcet Cycles occur if preferences are random

TABLE 4.1
PROBABILITY OF A CYCLICAL MAJORITY, $Pr(m,n)$

Number of Alternatives (m)	Number of Voters (n): 3	5	7	9	11	limit
3	.056	.069	.075	.078	.080	.088
4	.111	.139	.150	.156	.160	.176
5	.160	.200	.215			.251
6	.202					.315
limit	≈1.000	≈1.000	≈1.000	≈1.000	≈1.000	≈1.000

SOURCE: William H. Riker, *Liberalism Against Populism* (San Francisco: Freeman, 1982), p. 122

- The good news is that with few alternatives, it doesn't happen that often. The bad news is that with lots of alternatives, it happens for sure. In the real world, there are usually lots of alternatives.

Concluding Thoughts

- Even with perfect, honest democracy – which we have seen we don't always have in developing countries – there is no "right answer" in many cases
- Therefore who controls the agenda can matter a lot in practice
- Natural segway to our next section on dictators