14.661: Recitation 7

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Let's consider a perfectly competitive product market with demand curve $D(p)$. Producers have access to the CRTS production function $F(K, L)$. Since capital is not fixed here, we can think of this as a model of the long run. The question we want to ask is: What is the long-run elasticity of demand for labor at the market level? In particular, what parameters does this elasticity depend on? The derivation below will loosely follow Hamermesh's HOLE chapter on labor demand, though he leaves out some of the details.

1 Cost Minimization

Producers will choose their inputs to minimize the cost of producing output, whatever output level they settle on. This aspect of the production decision is described by the problem

$$c(w, r, q) = \min_{K,L} wL + rK$$

s.t.

$$F(K, L) \geq q$$

Note that this is EXACTLY like the EMP we learned about in consumer theory. We will now explore some properties of this problem.

1.1 Conditional Factor Demands

We call the solutions to the CMP, $K^c(w, r, q)$ and $L^c(w, r, q)$, conditional factor demands; they tell us the mix of inputs chosen for a given set of prices and target output level. The producer-theory version of Shephard's Lemma (just using the envelope theorem) is

$$\frac{\partial c(w, r, q)}{\partial w} = L^c(w, r, q),\ \frac{\partial c(w, r, q)}{\partial r} = K^c(w, r, q)$$

1.2 Homogeneity of the Cost Function

With CRTS, the cost function is homogeneous of degree 1 in input prices:

$$c(\alpha w, \alpha r, q) = \min_{K,L} \alpha wL + \alpha rK$$

s.t.

$$F(K,L) \geq q$$

which is

$$\alpha \min_{K,L} wL + rK$$

s.t.

$$F(K,L) \geq q$$

$$= \alpha c(w,r,q)$$

With CRTS, the cost function is also homogeneous of degree 1 in q:

$$c(w,r,\alpha q) = \min_{K,L} wL + rK$$

s.t.

$$F(K,L) \geq \alpha q$$

which is

$$c(w,r,\alpha q) = \alpha \min_{K,L} w\left(\frac{L}{\alpha}\right) + r\left(\frac{K}{\alpha}\right)$$

s.t.

$$F\left(\tfrac{K}{\alpha}, \tfrac{L}{\alpha}\right) \geq q$$

$$= \alpha c(w,r,q)$$

This means that for any q,

$$c(w,r,q) = c\left(w,r,q \cdot 1\right)$$

$$= q \cdot c(w,r,1)$$

$$\equiv q \cdot c^u(w,r)$$

where $c^u(w,r)$ is the **unit cost function**. This also implies that

$$c_w = qc^u_w,\ c_r = qc^u_r,\ c_{rw} = qc^u_{rw}$$

1.3 Elasticity of Substitution

The elasticity of substitution σ for a production function is defined by

$$\sigma \equiv -\frac{\partial \log\left({}^{L^c}\!/_{K^c}\right)}{\partial \log\left({}^{w}\!/_{r}\right)}$$

Intuitively, this is the proportionate reduction in a firm's labor/capital ratio in response to an increase in the relative price of labor, holding output constant.

Recall that the cost function c is homogeneous of degree 1 in factor prices, so by Euler's theorem its derivatives are homogenous of degree zero, and its second derivatives are homogeneous of degree -1:

$$c_r(w,r,q) = c_r\left(\tfrac{w}{r},1,q\right),\; c_w(w,r,q) = c_w\left(\tfrac{w}{r},1,q\right)$$

$$c_{ww}(w,r,q) = \frac{1}{r}c_{ww}\left(\tfrac{w}{r},1,q\right),\; c_{rw}(w,r,q) = \frac{1}{r}c_{rw}\left(\tfrac{w}{r},1,q\right)$$

Using Shephard's Lemma, we can write

$$\log\left({}^{K^c}\!/_{L^c}\right) = \log c_r(w,r,q) - \log c_w(w,r,q)$$
$$= \log c_r\left(\tfrac{w}{r},1,q\right) - \log c_w\left(\tfrac{w}{r},1,q\right)$$

and we have

$$\sigma = \frac{\partial \log\left({}^{K^c}\!/_{L^c}\right)}{\partial \log\left({}^{w}\!/_{r}\right)} = \frac{\partial \log\left({}^{K^c}\!/_{L^c}\right)}{\partial\left({}^{w}\!/_{r}\right)}\cdot\frac{w}{r}$$
$$= \frac{w}{r}\left(\frac{c_{rw}\left(\frac{w}{r},1,q\right)}{c_r\left(\frac{w}{r},1,q\right)} - \frac{c_{ww}\left(\frac{w}{r},1,q\right)}{c_w\left(\frac{w}{r},1,q\right)}\right)$$
$$= \frac{w}{r}\left(\frac{rc_{rw}}{c_r} - \frac{rc_{ww}}{c_w}\right)$$

Note further that by h.o.d. 0 of c_w in factor prices, $wc_{ww} + rc_{wr} = 0$, so

$$\frac{w}{r} = -\frac{c_{wr}}{c_{ww}} \text{ and } wc_{ww} = -rc_{wr}$$

Subbing these in gives

$$\sigma = -\frac{c_{wr}}{c_{ww}}\left(-\frac{wc_{ww}}{c_r} - \frac{rc_{ww}}{c_w}\right)$$
$$= \frac{c_{wr}}{c_{ww}}\left(\frac{wc_{ww}c_w + rc_{ww}c_r}{c_rc_w}\right)$$
$$= c_{wr}\left(\frac{wc_w + rc_r}{c_rc_w}\right)$$
$$\implies \sigma = \frac{c\cdot c_{wr}}{c_rc_w}$$

Evaluating this at $q = 1$,

$$\sigma = \frac{c^uc^u_{wr}}{c^u_rc^u_w}$$

This expression will be useful later on.

2 Market Demand

We are now in a position to derive some properties of the long-run market demand for labor.

2.1 Substitution and Scale Effects

The total market demand for labor is the sum over all firms in the industry:

$$L(w,r) = \sum_j L_j(w,r)$$

$$= \sum_j L_j^c(w,r,q_j)$$

$$= \sum_j c_w(w,r,q_j)$$

$$= \sum_j q_j c_w^u(w,r)$$

$$= Q c_w^u(w,r)$$

where Q is the total quantity produced in the market. In the background, the product market must be in equilibrium. In equilibrium, Q will have to be equal to $D(p)$, so that

$$L(w,r) = D(p) c_w^u(w,r)$$

Furthermore, in competitive equilibrium it must be the case that the product price is equal to marginal cost for each firm:

$$p = \frac{\partial c(w,r,q_j)}{\partial q_j} = \frac{\partial \left[q_j c^u(w,r) \right]}{\partial q_j} = c^u(w,r)$$

so

$$L(w,r) = D\left(c^u(w,r)\right) c_w^u(w,r)$$

We can now differentiate this expression to obtain the slope of labor demand:

$$\frac{\partial L}{\partial w} = D(p) \cdot c_{ww}^u + D'(p) \cdot \left(c_w^u\right)^2$$

These are the substitution and scale effects from class.

2.2 Fundamental Law of Factor Demand

It is customary to put our equation for the substitution and scale effects into elasticity terms. To do this, we just need to invoke some of the properties we proved in Section 1. Recall from 1.3 that

$$c^u_{ww} = -\frac{r}{w} c^u_{wr}$$

Plugging this into the labor demand expression gives

$$\frac{\partial L}{\partial w} = -D(p) \cdot \frac{r}{w} \cdot c^u_{wr} + D'(p) \cdot (c^u_w)^2$$

Also in 1.3, we showed that

$$\sigma = \frac{c^u c^u_{wr}}{c^u_r c^u_w} \implies c^u_{wr} = \sigma \cdot \frac{c^u_r c^u_w}{c^u}$$

so

$$\frac{\partial L}{\partial w} = -D(p) \cdot \frac{r}{w} \cdot \sigma \cdot \frac{c^u_r c^u_w}{c^u} + D'(p) \cdot (c^u_w)^2$$

In 2.1 we argued that

$$L = Qc^u_w, \ K = Qc^u_r$$

so we can plug in for these derivatives:

$$\frac{\partial L}{\partial w} = -D(p) \cdot \frac{r}{w} \cdot \frac{\sigma}{c^u} \cdot \frac{LK}{Q^2} + D'(p) \cdot \frac{L^2}{Q^2}$$

Using that $c^u = p$, $D(p) = Q$, and rearranging, this is

$$= -\frac{rK}{pQ} \cdot \frac{\sigma L}{w} + D'(p) \cdot \frac{L^2}{D(p)^2}$$

Now we can convert this to an elasticity:

$$\eta'_{LL} = \frac{\partial L}{\partial w} \cdot \frac{w}{L}$$

$$= -\frac{rK}{pQ} \cdot \frac{\sigma L}{w} \cdot \frac{w}{L} + D'(p) \cdot \frac{L^2}{D(p)^2} \cdot \frac{w}{L}$$

$$= -s_K \cdot \sigma + \frac{D'(p)}{D(p)} \cdot \frac{wL}{D(p)}$$

where $s_K = \dfrac{rK}{pQ}$, capital's share in total income from the product market. If we multiply and divide by p in the second term, this is

$$\eta'_{LL} = -s_K \cdot \sigma + D'(p) \cdot \frac{p}{D(p)} \cdot \frac{wL}{pD(p)}$$

$$= -s_K \cdot \sigma - \eta \cdot s_L,$$

where

$$\eta = |D'(p) \cdot \frac{p}{D(p)}|$$

is the absolute value of product demand, and $s_L = \dfrac{wL}{pQ}$. Finally, with competition there are no profits so we must have $s_K + s_L = 1$, so

$$\boxed{\eta'_{LL} = -(1 - s_L) \cdot \sigma - \eta \cdot s_L}$$

This is the Fundamental Law of Factor Demand; it tells us the long-run elasticity of the market demand curve for labor. We can immediately note some comparative statics:

1. $\sigma \uparrow \implies |\eta'_{LL}| \uparrow$: When substitution to other factors is easier, the market demand curve for labor is more elastic.
2. $\eta \uparrow \implies |\eta'_{LL}| \uparrow$: When product demand is more elastic, labor demand is more elastic.
3. $\left(s_L \uparrow \implies |\eta'_{LL}| \uparrow\right) \iff (\sigma < \eta)$: When labor's share of income increases, labor demand becomes more elastic iff the elasticity of product demand is larger than the elasticity of substitution.

These are three of the "Hicks-Marshall Rules."