

14.661 Recitation 6: Old-School Micro

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Today we're going to cover some classic topics from consumer theory. In particular, we want to examine what it means for goods to be substitutes or complements in the most rigorous sense.

1 Properties of Hicksian Demand

1.1 Background

Recall the Expenditure Minimization Problem:

$$\begin{aligned} e(p, \bar{u}) &= \min_x p \cdot x \\ &\text{s.t.} \\ u(x) &\geq \bar{u} \end{aligned}$$

The consumption vector that solves this problem is $x^c(p, \bar{u})$, the Hicksian (or “compensated”) demand function. The minimized minimand, $e(p, \bar{u})$, is the expenditure function. The derivatives $S_{ij} = \frac{\partial h_i}{\partial p_j}$ are compensated price responses; as we have seen, these are intimately related to substitution.

In an earlier recitation, we derived 2 properties of the expenditure function:

1. **Shephard's Lemma:** $\frac{\partial e(p, \bar{u})}{\partial p_i} = x_i^c$. This is a consequence of the envelope theorem.
2. **Concavity:** $e(p, \bar{u})$ is concave in prices. That is, $e(\alpha p_1 + (1 - \alpha)p_0, \bar{u}) \geq \alpha e(p_1, \bar{u}) + (1 - \alpha)e(p_0, \bar{u})$. Intuitively, this is a consequence of substitution.

We will use these in the proofs below.

1.2 The Slutsky Substitution Matrix

The compensated price responses S_{ij} tell us how demand changes as we change prices, compensating the agent to keep him on the same indifference curve. These are pure substitution effects. To analyze these derivatives we usually organize them into the matrix

$$S = \begin{bmatrix} S_{11} & S_{12} & \cdots & S_{1N} \\ S_{21} & S_{22} & \cdots & \vdots \\ \vdots & \cdots & \ddots & \vdots \\ S_{N1} & \cdots & \cdots & S_{NN} \end{bmatrix}$$

The notable properties of the Slutsky matrix are:

1. Symmetry: $S_{ij} = S_{ji}$.

This property is a direct consequence of Shephard's Lemma. Since S_{ij} is the cross-partial derivative of the expenditure function, and the order of differentiation doesn't matter, it must be equal to S_{ji} :

$$S_{ij} = \frac{\partial x_i^c}{\partial p_j} = \frac{\partial}{\partial p_j} \left[\frac{\partial e(p, \bar{u})}{\partial p_i} \right] = \frac{\partial^2 e(p, \bar{u})}{\partial p_i \partial p_j} = S_{ji}$$

2. Negative Semi-Definiteness: For any vector t and any prices p , $t'S(p)t \leq 0$.

This is a consequence of concavity of the expenditure function. Since $x^c(p, \bar{u}) = \nabla_p e(p, \bar{u})$, the Slutsky matrix is the matrix of second derivatives of the expenditure function with respect to price:

$$S = \nabla_p^2 e(p, \bar{u})$$

If a function is concave and differentiable, the second derivative matrix is necessarily negative semi-definite. Because deriving this result directly is informative, I provide a direct proof at the end of these notes.

This property is actually quite intuitive. One consequence of negative semi-definiteness is that the determinants of the principle submatrices must alternate in sign, starting with a negative. In the 2×2 case this means that

- $S_{11} < 0$: Own-price compensated price responses are negative. In other words, if the price of a good increases, the substitution effect leads to decreased consumption of that good. Since the order of goods in the matrix is arbitrary, this implies that $S_{ii} < 0 \forall i$.
- $S_{11}S_{22} - S_{12}^2 > 0$: Own-price substitution effects are large enough to "swamp" cross-price effects in this specific sense.

1.3 Homogeneity

Other noteworthy properties of Hicksian Demand involve homogeneity. To derive these, we first need a couple of definitions and results that you've probably seen before:

1. A function $f(z)$, where z is a vector, is **homogeneous of degree k** if $f(\alpha z) = \alpha^k f(z)$ for any α .
2. If $f(z)$ is homogeneous of degree k , then ∇f is homogeneous of degree $(k - 1)$.
3. **Euler's Theorem:** If $f(z)$ is homogeneous of degree k , then $\sum_j \frac{\partial f}{\partial z_j} \cdot z_j = kf(z)$

To apply these to Hicksian Demand, note that

$$\begin{aligned} e(\alpha p, \bar{u}) &= \min_x \alpha p \cdot x \text{ s.t. } u(x) \geq \bar{u} \\ &= \alpha \min_x p \cdot x \text{ s.t. } u(x) \geq \bar{u} \\ &= \alpha e(p, \bar{u}), \end{aligned}$$

so **the expenditure function is h.o.d. 1 in prices**. Intuitively, if all prices are doubled, it costs me exactly twice as much to get to a given utility level.

Since $x^c = \nabla e$ by Shephard's Lemma, it immediately follows that **Hicksian demands are h.o.d. 0 in prices**. This is also quite intuitive; if all prices are scaled up by the same factor, it costs me more in nominal terms to get to a given utility level, but the goods I buy to get there don't change since *relative* prices are still the same.

If we apply Euler's theorem to the Hicksian demand function for good i , we have that

$$\sum_j \frac{\partial x_i^c}{\partial p_j} \cdot p_j = 0$$

That is,

$$\sum_j S_{ij} p_j = 0.$$

2 Substitutes and Complements

2.1 Definition

We are now in a position to think about substitutes and complements:

Goods i and j are **Hicksian (p-)complements** if $S_{ij} < 0$

Goods i and j are **Hicksian (p-)substitutes** if $S_{ij} > 0$

Intuitively, i and j are substitutes if an increase in the price of good j leads me to consume more i ; I substitute away from j towards i . Note that these definitions are only about substitution. Since we're talking about Hicksian demands, there are no income effects.

2.2 Two-good case

If there are only two goods, then the Euler's Theorem result from above says that

$$p_1 \frac{\partial x_1^c}{\partial p_2} + p_2 \frac{\partial x_2^c}{\partial p_2} = 0$$

$$\implies S_{12} = \frac{\partial x_1^c}{\partial p_2} = -\frac{p_2}{p_1} \cdot \frac{\partial x_2^c}{\partial p_2} > 0$$

That is, when there are only two goods, they are **necessarily substitutes**. This makes sense – if the price of good 2 increases, the compensated demand for good 2 must fall since own-price substitution effects are negative. But by definition of compensated demand, the consumer must end up with the same utility as before. With two goods, the only way to get back the lost utility coming from the reduction in x_2 is to increase x_1 . This is also clear graphically; there is no way to draw a movement along an indifference curve in response to an increase in the price of good 2 that results in less consumption of good 1.

This analysis shows that when we talk about complements in the 2-good scenario, we are really talking about cases where the negative income effect for good j overwhelms the positive substitution effect when the price of i goes up. In some sense goods like this are “weak” substitutes, but using the Hicksian definition they are still substitutes and not complements.

The classic 2-good example of “complements” is the Leontief utility function;

$$u(x_1, x_2) = \min \{ax_1, bx_2\}$$

With these preferences the consumer purchases x_1 and x_2 only in fixed proportions: $x_2 = \frac{a}{b}x_1$. Note, however, that if we draw this in 2-d you can see that if prices change and the consumer stays on the same indifference curve, consumption does not change **at all**. This is as close to Hicksian complementarity as we can get with 2 goods.

2.3 Preferences and Substitutes

We often work with utility functions that actually implicitly impose that all goods are substitutes in the Hicksian sense. In particular, for utility functions of the form

$$u(x) = F(v_1(x_1) + \dots + v_N(x_N))$$

with F increasing and the v 's concave, all goods must be substitutes. Many common preference structures are of this form; for example, consider the Stone-Geary Preferences

$$u(x) = \prod_i (x_i - \gamma_i)^{\alpha_i}, \quad x_i > \gamma_i$$

This is of the additive form discussed above, with

$$F(\cdot) = \exp(\cdot), \text{ and } v_i(x_i) = \alpha_i \log(x_i - \gamma_i)$$

So with Stone-Geary preferences, it must be the case that all goods are substitutes even with more than 2.

To derive this result for the general case, note that the FOCs determining the choice of good i is

$$F'(\cdot)v'_i(x_i) = \lambda p_i$$

so

$$\frac{v'_i(x_i)}{v'_1(x_1)} = \frac{p_i}{p_1}$$

$$\implies x_i = v_i'^{-1} \left(v_1'(x_1) \cdot \frac{p_i}{p_1} \right) \quad \forall i$$

Hicksian demands satisfy

$$u(x^c) = \bar{u}$$

$$\implies \bar{u} = F \left(v_1(x_1^c) + \sum_{i \neq 1} v_i \left(v_i'^{-1} \left(v_1'(x_1) \cdot \frac{p_i}{p_1} \right) \right) \right)$$

Let's differentiate this object w.r.t. p_2 :

$$0 = F' \cdot \left[v_1' \cdot \frac{dx_1^c}{dp_2} + \sum_{i \neq 1} \left\{ v_i' \cdot (v_i'^{-1})' \cdot v_1'' \cdot \frac{dx_1^c}{dp_2} \cdot \frac{p_i}{p_1} \right\} + v_2' \cdot (v_2'^{-1})' \cdot v_1' \cdot \frac{1}{p_1} \right]$$

$$\implies \frac{dx_1^c}{dp_2} = -\frac{v_2' \cdot (v_2'^{-1})' \cdot \frac{v_1'}{p_1}}{v_1' + \sum_{i \neq 1} \left\{ v_i' \cdot (v_i'^{-1})' \cdot v_1'' \cdot \frac{p_i}{p_1} \right\}} > 0$$

Under our assumptions, $v_i' > 0$, and the v_i are concave so v_i' (and therefore $v_i'^{-1}$) are decreasing so v_i'' , $(v_i'^{-1})' < 0$. This whole expression is therefore positive. Since the choice of goods 1 and 2 was arbitrary, this works for any pair, and all goods are substitutes.

2.4 Complements

The above result shows that for a wide class of preferences, all goods must be substitutes. So what do preferences look like for complements? We know that there must be at least 3 goods. Consider the preferences

$$U(x, y, z) = x^\alpha (\min\{y, z\})^{1-\alpha}$$

It is clear that with these preferences, y and z will be complements. The consumer will always consume y and z in fixed proportions; to do anything else would be to waste money that could be spent on x . What happens when p_z goes up? Intuitively, the consumer will want to shift consumption away from z and towards x since z and x are substitutes. But reducing consumption of z means reducing consumption of y ; if she buys less z , then she has no need for some of her y , so we should expect $\frac{dy}{dp_z} < 0$. To see this mathematically, note that since $y = z$ at the optimum, we can up the EMP as

$$\min p_x x + (p_y + p_z)y$$

s.t.

$$x^\alpha y^{1-\alpha} \geq \bar{u}$$

By the usual FOC, we'll have

$$\frac{p_x}{p_y + p_z} = \frac{\alpha}{1 - \alpha} \cdot \frac{y}{x}$$

$$\implies x = \frac{p_y + p_z}{p_x} \cdot \frac{\alpha}{1 - \alpha} \cdot y$$

$$\implies \left(\frac{p_y + p_z}{p_x} \cdot \frac{\alpha}{1 - \alpha} \right)^\alpha y = \bar{u}$$

$$\implies y = \bar{u} \cdot \left(\frac{\alpha}{1 - \alpha} \cdot \frac{p_x}{p_y + p_z} \right)^\alpha$$

$$\implies \frac{dy}{dp_z} = -\bar{u} \alpha \left(\frac{\alpha}{1 - \alpha} \cdot \frac{p_x}{p_y + p_z} \right)^{\alpha-1} \cdot \frac{\alpha}{1 - \alpha} \cdot \frac{p_x}{(p_y + p_z)^2} < 0$$

so these are complements.

More generally, if the utility function is

$$u(x, y, z) = x^\alpha (y^\rho + z^\rho)^{\frac{1-\alpha}{\rho}},$$

y and z are complements as long as

$$\frac{1}{1-\rho} < \alpha$$

The substitutability over y and z is a constant elasticity of substitution (CES) function; the elasticity of substitution is $\sigma \equiv \frac{1}{1-\rho}$.

Appendix: Direct Proof of Slutsky Negative Semi-Definiteness

Let $h(p, u)$ be the Hicksian demand vector, where p is the price vector. The expenditure function is

$$e(p, u) = \sum p_i h_i = p \cdot h(p, u)$$

where the last equality uses the dot product. The Slutsky substitution matrix is defined as

$$S = D_p h(p, u) = \begin{bmatrix} \frac{\partial h_1}{\partial p_1} & \frac{\partial h_1}{\partial p_2} & \cdots & \frac{\partial h_1}{\partial p_N} \\ \frac{\partial h_2}{\partial p_1} & \frac{\partial h_2}{\partial p_2} & \cdots & \vdots \\ \vdots & \cdots & \ddots & \vdots \\ \frac{\partial h_N}{\partial p_1} & \cdots & \cdots & \frac{\partial h_N}{\partial p_N} \end{bmatrix}$$

where D_p denotes a matrix derivative. We want to prove that S is negative semi-definite:

$$\text{For any vector } t, t' S t \leq 0$$

To prove this, we will first prove the following lemma:

Lemma (Compensated Law of Demand):

$$\text{For any pair of price vectors } p' \text{ and } p \text{ and any utility level } u, (p' - p) \cdot (h(p', u) - h(p, u)) \leq 0$$

Proof:

Note that

$$\begin{aligned} (p' - p) \cdot (h(p', u) - h(p, u)) &= p' \cdot h(p', u) - p' \cdot h(p, u) + p \cdot h(p, u) - p \cdot h(p', u) \\ &= [e(p', u) - p' \cdot h(p, u)] - [e(p, u) - p \cdot h(p', u)] \end{aligned}$$

Since $h(p, u)$ yields utility u , it must be more expensive than $e(p', u)$ at prices p' , so the first term is negative. The same argument holds for the second term, so the whole quantity is negative and we have the lemma.

Now, we can move on to proving negative semi-definiteness of the Slutsky matrix.

Proof of negative semi-definiteness:

Consider arbitrary price vectors p' and p . Define

$$v = p' - p$$

$$p(t) = p + t \cdot v = p + tp' - tp$$

where t is a scalar. Now define the following function of t :

$$g(t) = v \cdot (h(p(t)) - h(p))$$

Here I've suppressed dependence on u and written $h(p, u) = h(p)$; the following works for any u . Note that

$$\begin{aligned} g'(t) &= v' D_p h(p(t)) \cdot (p' - p) \\ &= v' S(p(t)) v \end{aligned}$$

Here I've just used properties of matrix derivatives. Next consider what happens around 0:

$$p(0) = p + 0 \cdot v = p$$

so

$$g(0) = (p' - p) \cdot (h(p) - h(p)) = 0$$

Next, note that for $t \neq 0$,

$$\begin{aligned} v &= p' - p = \frac{1}{t}(tp' - tp) \\ &= \frac{1}{t}(p(t) - p) \end{aligned}$$

so we can write

$$g(t) = \frac{1}{t}(p(t) - p) \cdot (h(p(t)) - h(p))$$

By the compensated law of demand, we therefore know that $g(t) \leq 0$ for any $t > 0$. Since $g(0) = 0$ and $g(t) \leq 0$ for $t > 0$, we must have $g'(0) \leq 0$; if $g'(0)$ was positive, then for small enough t we would have $g(t)$ positive, which we know isn't true. But using the expression above, we know that

$$\begin{aligned} g'(0) &= v' S(p(0)) v \\ &= v' S(p) v \leq 0 \end{aligned}$$

Since p and p' were arbitrary, we can vary p' to make v whatever vector we want, so we've proved that for any v

$$v' S(p) v \leq 0$$

which means that the Slutsky matrix S is negative semi-definite.

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