

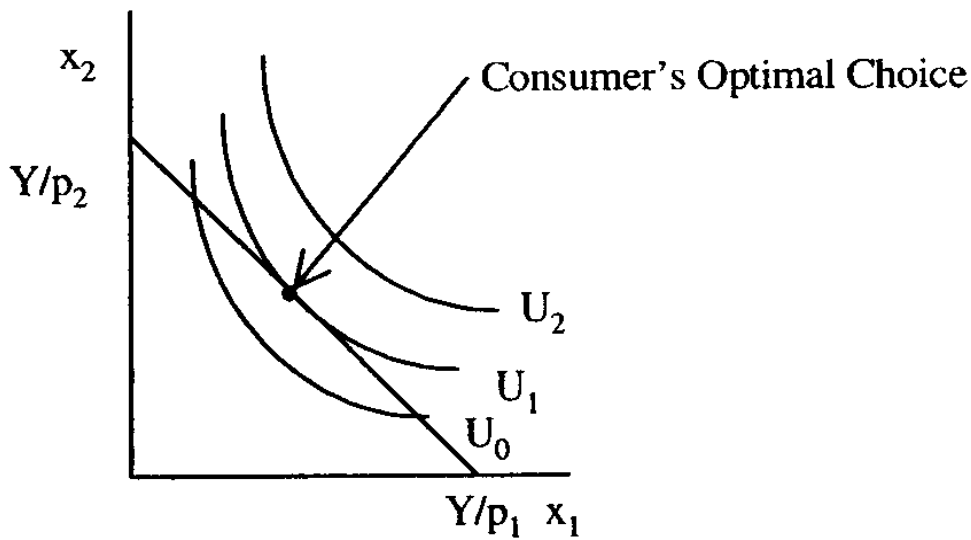
14.64 Review Exercise Solutions

Labor Economics  
Spring 2009

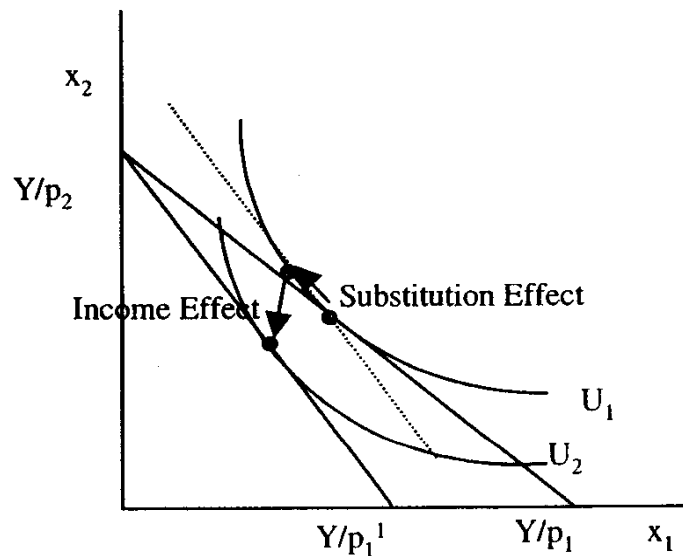
Joshua D. Angrist  
Amanda Pallais

A. A consumer chooses between two goods,  $x_1$  and  $x_2$ , with prices  $p_1$  and  $p_2$  given income  $y$ , so as to maximize utility.

1. Graph the consumer's problem in  $x_1$  and  $x_2$  space. Show the consumer's optimal choice.



2. Give a graphical decomposition of the income and substitution effects of a change in  $p_1$  on the demand for  $x_1$ .



B. Assume the consumer's utility function is given by:

$$u(x_1, x_2) = \alpha \ln(x_1 - \gamma_1) + (1 - \alpha) \ln(x_2 - \gamma_2),$$

where:  $0 < \alpha < 1$ ,  $x_1 > \gamma_1 > 0$ ,  $x_2 > \gamma_2 > 0$ .

(This is called a Stone-Geary utility function. The parameters  $x_1$  and  $x_2$  are sometimes thought of as "subsistence" levels of consumption, below which utility is not defined.)

1. What is the marginal rate of substitution (MRS) for this utility function?

By definition, the  $MRS = \frac{-dx_2}{dx_1} \Big|_{U=\bar{U}}$

$$MRS = \frac{-dx_2}{dx_1} \Big|_{U=\bar{U}} = \frac{\partial U(x_1, x_2) / \partial x_1}{\partial U(x_1, x_2) / \partial x_2}$$

$$\partial U(x_1, x_2) / \partial x_1 = \frac{\alpha}{x_1 - \gamma_1}$$

$$\partial U(x_1, x_2) / \partial x_2 = \frac{1 - \alpha}{x_2 - \gamma_2}$$

$$MRS = \frac{\partial U(x_1, x_2) / \partial x_1}{\partial U(x_1, x_2) / \partial x_2} = \left( \frac{\alpha}{1 - \alpha} \right) \left( \frac{x_2 - \gamma_2}{x_1 - \gamma_1} \right)$$

2. Present the first-order conditions for optimal choices of  $x_1$  and  $x_2$  as a single equation involving the MRS. What is the graphical interpretation of this condition?

The first order condition is

$$MRS = \left( \frac{\alpha}{1 - \alpha} \right) \left( \frac{x_2^* - \gamma_2}{x_1^* - \gamma_1} \right) = \frac{p_1}{p_2}$$

This means that the additive inverse of the slope of the indifference curve (the MRS) must equal the additive inverse of the slope of the budget line (the price ratio).

3. Derive the uncompensated ("Marshallian") demand function for  $x_1$  for this consumer as a function of prices and income.

$$\max_{x_1, x_2} \alpha \ln(x_1 - \gamma_1) + (1 - \alpha) \ln(x_2 - \gamma_2) \quad \text{subject to } p_1 x_1 + p_2 x_2 = y$$

$$L = \alpha \ln(x_1 - \gamma_1) + (1 - \alpha) \ln(x_2 - \gamma_2) + \lambda (y - p_1 x_1 + p_2 x_2)$$

$$(1) \frac{\partial L}{\partial x_1} = \frac{\alpha}{x_1^* - \gamma_1} - \lambda p_1 = 0$$

$$(2) \frac{\partial L}{\partial x_2} = \frac{1 - \alpha}{x_2^* - \gamma_2} - \lambda p_2 = 0$$

$$(3) \frac{\partial L}{\partial \lambda} = y - p_1 x_1^* + p_2 x_2^* = 0$$

Since we must consume positive amounts of both goods ( $x_1 > \gamma_1 > 0$  and  $x_2 > \gamma_2 > 0$ ), (1) and (2) must hold with equality. Since the marginal utility of both goods is always positive,  $\lambda > 0$  so that (3) also holds with equality.

$$\text{Rewriting (3) produces } x_2^* = \frac{y - p_1 x_1^*}{p_2}$$

Rewriting and combining (1) and (2)

$$\frac{\alpha}{x_1^* - \gamma_1} = \lambda p_1$$

$$\frac{1 - \alpha}{x_2^* - \gamma_2} = \lambda p_2$$

$$\left(\frac{\alpha}{1 - \alpha}\right) \left(\frac{x_2^* - \gamma_2}{x_1^* - \gamma_1}\right) = \frac{\lambda p_1}{\lambda p_2} = \frac{p_1}{p_2}$$

Substituting in for  $x_2$  gives

$$\left(\frac{\alpha}{1 - \alpha}\right) \left(\frac{y - p_1 x_1 - p_2 \gamma_2}{x_1 - \gamma_1}\right) = \frac{p_1}{p_2}$$

$$\left(\frac{\alpha}{1 - \alpha}\right) \left(\frac{y - p_1 x_1^* - p_2 \gamma_2}{x_1^* - \gamma_1}\right) = p_1$$

$$\left(\frac{\alpha}{1 - \alpha}\right) (y - p_1 x_1^* - p_2 \gamma_2) = p_1 (x_1^* - \gamma_1)$$

$$\left(\frac{\alpha}{1 - \alpha}\right) (y - p_2 \gamma_2) + p_1 \gamma_1 = \frac{p_1}{1 - \alpha} x_1^*$$

$$\frac{\alpha(y - p_2 \gamma_2)}{p_1} + (1 - \alpha) \gamma_1 = x_1^*$$

$$x_1^*(p_1, p_2, y) = \frac{\alpha(y - p_2 \gamma_2)}{p_1} + (1 - \alpha) \gamma_1$$

4. Derive the compensated ("Hicksian") demand function for  $x_1$  for this consumer as a function of  $p_1$ ,  $p_2$  and a fixed level of utility,  $u^*$ . Derive the same relationship graphically. What is the connection between the compensated demand function and the substitution effects of question A.2?

$$\min_{x_1, x_2} p_1 x_1 + p_2 x_2 \quad \text{s.t.} \quad \alpha \ln(x_1 - \gamma_1) + (1 - \alpha) \ln(x_2 - \gamma_2) = u^*$$

$$L = p_1 x_1 + p_2 x_2 + \lambda (u^* - \alpha \ln(x_1 - \gamma_1) + (1 - \alpha) \ln(x_2 - \gamma_2))$$

$$(1) \quad \frac{\partial L}{\partial x_1} = p_1 - \frac{\lambda \alpha}{x_1^* - \gamma_1} = 0$$

$$(2) \quad \frac{\partial L}{\partial x_2} = p_2 - \lambda \left(\frac{1 - \alpha}{x_2^* - \gamma_2}\right) = 0$$

$$(3) \quad \frac{\partial L}{\partial \lambda} = u^* - \alpha \ln(x_1^* - \gamma_1) + (1 - \alpha) \ln(x_2^* - \gamma_2) = 0$$

Rearranging and combining (1) and (2) gives

$$p_1 = \frac{\lambda \alpha}{x_1^* - \gamma_1}$$

$$p_2 = \lambda \left(\frac{1 - \alpha}{x_2^* - \gamma_2}\right)$$

$$\frac{p_1}{p_2} = \left(\frac{\alpha}{1 - \alpha}\right) \left(\frac{x_2^* - \gamma_2}{x_1^* - \gamma_1}\right)$$

$$x_2^* - \gamma_2 = \frac{p_1}{p_2} \left(\frac{1 - \alpha}{\alpha}\right) (x_1^* - \gamma_1)$$

Substituting in for  $x_2^* - \gamma_2$  in (3) gives

$$u^* = \alpha \ln(x_1^* - \gamma_1) + (1 - \alpha) \ln\left[\frac{p_1}{p_2} \left(\frac{1 - \alpha}{\alpha}\right) (x_1^* - \gamma_1)\right]$$

$$u^* = \ln(x_1^* - \gamma_1) + (1 - \alpha) \ln\left[\frac{p_1}{p_2} \left(\frac{1 - \alpha}{\alpha}\right)\right]$$

$$e^{u^*} = (x_1^* - \gamma_1) \left(\frac{p_1}{p_2}\right)^{1 - \alpha} \left(\frac{1 - \alpha}{\alpha}\right)^{1 - \alpha}$$

$$x_1^* - \gamma_1 = e^{u^*} \left(\frac{\alpha p_2}{(1 - \alpha) p_1}\right)^{1 - \alpha}$$

$$x_1^*(p_1, p_2, u^*) = e^{u^*} \left(\frac{\alpha p_2}{(1 - \alpha) p_1}\right)^{1 - \alpha} + \gamma_1$$

The change in the uncompensated demand for  $x_1$  when  $p_1$  changes (keeping the utility level  $u^*$  constant) is the same as the change in the demand for  $x_1$  due only to the substitution effect in question A.2.

C. Applied researchers often work with the Cobb-Douglas production function:

$$Q = \gamma L^\alpha K^\beta; \alpha > 0, \beta > 0, \gamma > 0,$$

where L is Labor, K is capital, and Q is the quantity of output.

1. What must be true for this production function to exhibit constant returns to scale?

For this function to have constant returns to scale, multiplying the inputs L and K by any factor  $\lambda$ , must also multiply the final output, Q, by the same factor  $\lambda$ . That is

$$\gamma(\lambda L)^\alpha (\lambda K)^\beta = \lambda Q$$

$$\gamma \lambda^\alpha L^\alpha \lambda^\beta K^\beta = \lambda Q$$

$$(\lambda^{\alpha+\beta}) \gamma L^\alpha K^\beta = \lambda Q$$

$$(\lambda^{\alpha+\beta}) Q = \lambda Q$$

Since  $\lambda$  can be any positive number, this implies that

$$\alpha + \beta = 1$$

2. Econometricians have estimated  $\alpha$  and  $\beta$  by fitting the following equation to time series data on measures of output, capital, and labor:

$$\ln Q_t = \ln \gamma + \alpha \ln L_t + \beta \ln K_t + \varepsilon_t,$$

where  $\varepsilon_t$  is an error term and t indexes time series observations.

- (i) Where does this equation come from?

It comes from taking the log of both sides of the Cobb-Douglas production function and then adding an error term.

- (ii) Assuming you have the necessary data, describe two ways to construct a statistical test for constant returns to scale in production.

- (1) Estimate the equation  $\ln Q_t = \ln \gamma + \alpha \ln L_t + \beta \ln K_t + \varepsilon_t$ , obtaining estimates  $\hat{\alpha}$  and  $\hat{\beta}$ .

Then create a t-statistic  $t = \frac{\hat{\alpha} + \hat{\beta} - 1}{\sqrt{\hat{\sigma}_\alpha^2 + \hat{\sigma}_\beta^2 + 2\hat{\sigma}_{\alpha\beta}}}$  to test the hypothesis that  $\alpha + \beta = 1$ .

Where  $\hat{\sigma}_\alpha^2 + \hat{\sigma}_\beta^2 + \hat{\sigma}_{\alpha\beta}$  is the estimated variance of  $\hat{\alpha} + \hat{\beta}$  and comes from the fact that  $\text{Var}(a+b) = \text{Var}(a) + \text{Var}(b) + 2\text{Cov}(a,b)$ .

Under the null hypothesis,  $t \sim t(n-k)$  where n is the sample size and k is the number of parameters

estimated. Here  $k=3$ .

- (2) Estimate the equation  $\ln Q_t = \ln \gamma + \alpha \ln L_t + \beta \ln K_t + \varepsilon_t$  as before. Since we are not restricting  $\alpha + \beta = 1$ , this is the “unrestricted” regression. Obtain the sum of squared residuals and call this  $SSR_{UR}$  where “UR” stands for unrestricted.

Now restrict  $\alpha + \beta = 1$  so that  $\alpha = 1 - \beta$ .

$$\begin{aligned}\ln Q_t &= \ln \gamma + (1 - \beta) \ln L_t + \beta \ln K_t + \varepsilon_t \\ \ln Q_t - \ln L_t &= \ln \gamma - \beta \ln L_t + \beta \ln K_t + \varepsilon_t \\ \ln(Q_t/L_t) &= \ln \gamma + \beta \ln(K_t/L_t) + \varepsilon_t\end{aligned}$$

Estimate this equation, the “restricted” regression and obtain the sum of squared residuals  $SSR_R$ .

Perform an F test: 
$$F = \frac{(SSR_R - SSR_{UR})/m}{SSR_{UR}/(n-k)} \sim F(m, n-k)$$

Here  $n$  is the number of observations,  $m$  is the number of linear restrictions ( $m=1$ ) and  $k$  is the number of parameters to be estimated in the unrestricted model ( $k=3$ ).

D. Use a graph to explain the relationship between the purchase of insurance and the concept of risk aversion. Why might someone who purchases fire insurance also play the lottery?

See the attached graphs.

Graph 1 shows that the individual would prefer receiving wealth  $\frac{1}{2} w_1 + \frac{1}{2} w_2$  with certainty than entering a lottery where 50% of the time she received wealth  $w_1$  and the rest of the time she received wealth  $w_2$ . If she takes  $\frac{1}{2} w_1 + \frac{1}{2} w_2$  with certainty, she receives utility  $u(\frac{1}{2} w_1 + \frac{1}{2} w_2)$ . If she takes the lottery, her expected value is  $u(\frac{1}{2} w_1 + \frac{1}{2} w_2)$ . We can see from the graph that  $u(\frac{1}{2} w_1 + \frac{1}{2} w_2) > u(\frac{1}{2} w_1 + \frac{1}{2} w_2)$ .

The expected utility from a lottery with two outcomes is a point on the straight line connecting the utility levels at the two outcomes. When an individual's utility function is concave, this line will always lie below the utility curve, indicating that the individual would prefer to have the expected value of the lottery for certain than enter the lottery: she is risk averse.

An individual's utility function does not need to be globally concave or convex. Thus, the individual can exhibit risk aversion in some parts of her utility function and risk-loving behavior in others. For an example, see the second graph.

Question D

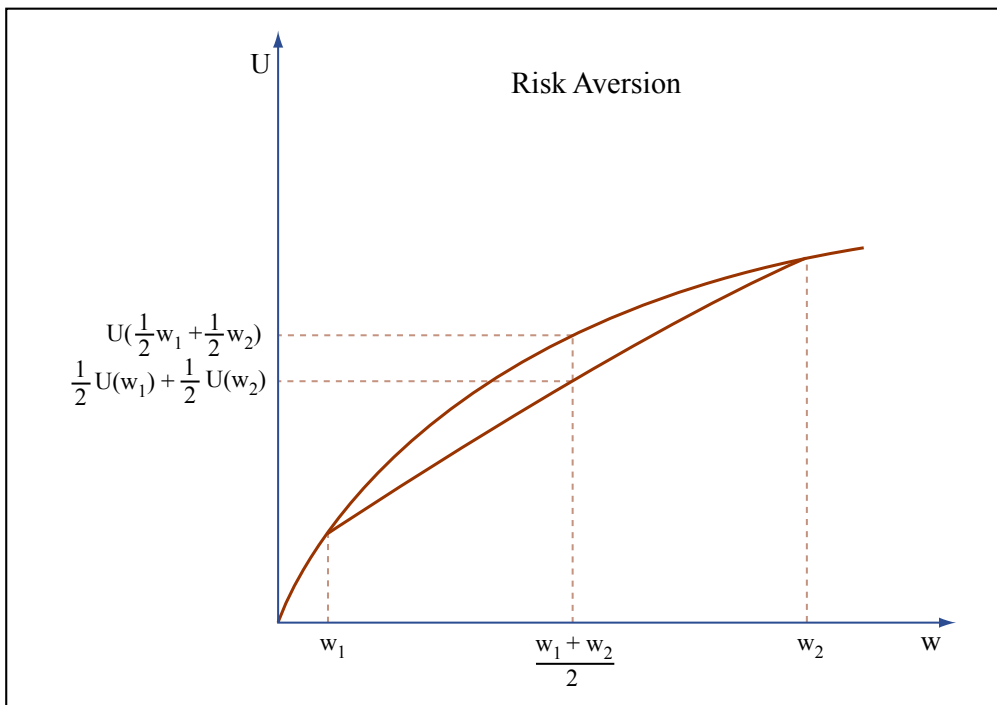


Figure by MIT OpenCourseWare.

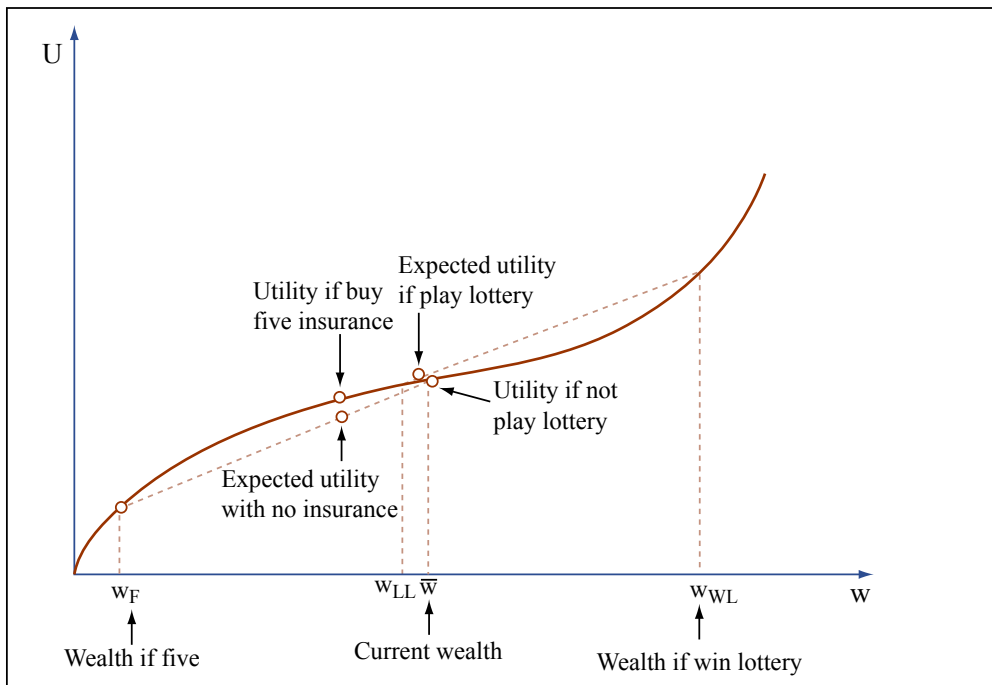


Figure by MIT OpenCourseWare.

E. Your uncle gives you a government bond for your Bar Mitzvah that can be given back to the government for 100 dollars in five years. You would rather have cash. How much could you sell the bond for today if the interest rate on FDIC insured savings accounts is fixed at 5 percent (compounded annually)?

You would sell your bond for some amount  $X$  such that  $X$ , invested in a vehicle that returns 5% per year and is compounded annually, is worth \$100 in five years. If  $\$X$  is invested, at the end of the first year it is worth  $\$X(1.05)$ . That amount then earns 5% interest the following year, and so on – so that by the end of five years,  $\$X$  is worth  $\$X(1.05)^5$ . Equating to \$100:  $X = \frac{100}{(1.05)^5} = \$78.35$ . \$78.35, invested such that it returns 5% per year, will be worth \$100 in five years.

F. Suppose that the demand for MP3 downloads is a linear function of their price and the annual income of college students. Make up some notation for a demand equation that expresses this model of demand (assume downloads are not free). Use this equation to compute the price-elasticity of demand for MP3 downloads as a function of price and income.

Let  $X$ =demand for MP3s. Suppose  $X = \alpha + \beta \cdot price + \gamma \cdot income$

Elasticity of demand with respect to price (P) is:

$$\frac{dX/dP}{X/P} = \beta \frac{price}{X} = \beta \frac{price}{\alpha + \beta \cdot price + \gamma \cdot income}.$$

G. Do private schools do a better job of preparing students for MIT than public schools? Consider two regressions that address this question, one short and one long.

The short regression looks like this:

$$MITGPA_i = \alpha_0 + \beta_0 PRIVATE_i + \varepsilon_{0i}$$

The long regression adds controls for a student's SAT score:

$$MITGPA_i = \alpha_1 + \beta_1 PRIVATE_i + \gamma_1 SAT_i + \varepsilon_{1i}$$

- (i) Why is the long regression likely to be a better measure of the effect of a private school education on MIT GPA?

Students who are more capable at age 14 may be more likely to attend private high schools. If this is the case, we might expect students who emerge from private high schools to be more capable when they enter MIT – and thus get higher grades -- even if there is no difference in the preparation private and public high schools provide. Controlling for SAT scores may control for the ability of the student, allowing us to estimate the effect of private high schools.

On the other hand, we might believe that if private high schools prepare students better for MIT, they may

also prepare students better for the SAT, so that controlling for SAT scores may lead us to underestimate the effect of private schools on MIT grades. Ideally, we would want to control for a measure of ability that the high school cannot affect. For example, we might want to control for 8<sup>th</sup> grade test scores.

- (ii) Use the omitted variables bias formula to give a precise description of the likely relationship between  $\beta_0$  and  $\beta_1$ .

$\beta_0$  is the bivariate regression coefficient from a regression of MITGPA on PRIVATE and a constant. This means

$$\beta_0 = \frac{\text{Cov}(\text{PRIVATE}_i, \text{MITGPA}_i)}{\text{Var}(\text{PRIVATE}_i)}$$

Substituting for MITGPA<sub>i</sub> gives

$$\beta_0 = \frac{\text{Cov}(\text{PRIVATE}_i, \alpha_1 + \beta_1 \text{PRIVATE}_i + \gamma_1 \text{SAT}_i + \varepsilon_{1i})}{\text{Var}(\text{PRIVATE}_i)}$$

$$\beta_0 = \frac{\beta_1 \text{Cov}(\text{PRIVATE}_i, \text{PRIVATE}_i) + \gamma_1 \text{Cov}(\text{PRIVATE}_i, \text{SAT}_i) + \text{Cov}(\text{PRIVATE}_i, \varepsilon_{1i})}{\text{Var}(\text{PRIVATE}_i)}$$

We know that by definition  $\text{Cov}(\text{PRIVATE}_i, \text{PRIVATE}_i) = \text{Var}(\text{PRIVATE}_i)$ .

By the OLS assumptions, we also know that  $\varepsilon_{1i}$  must be uncorrelated with the independent variables in the long regression, so  $\text{Cov}(\text{PRIVATE}_i, \varepsilon_{1i}) = 0$ .

Substituting these in gives

$$\beta_0 = \beta_1 + \gamma_1 \frac{\text{Cov}(\text{PRIVATE}_i, \text{SAT}_i)}{\text{Var}(\text{PRIVATE}_i)}$$

We recognize that  $\frac{\text{Cov}(\text{PRIVATE}_i, \text{SAT}_i)}{\text{Var}(\text{PRIVATE}_i)}$  is just the regression coefficient from a

bivariate regression of SAT on PRIVATE. Thus,

$$\beta_0 = \beta_1 + \gamma_1 \delta_1$$

where  $\text{SAT}_i = \delta_1 + \delta_1 \text{PRIVATE}_i + \varepsilon_{2i}$ .



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