14.462 Advanced Macroeconomics
Spring 2004

Midterm Exam Solution

1. The first order condition for consumption is

$$e^{-bc_i} = \mu p_i$$

and solving for consumption yields

$$c_i = -\frac{1}{b}\left[\log(p_i) + \log(\mu)\right].$$

Substituting into the budget constraint yields

$$-\frac{1}{b}\log(\mu) = \bar{c}$$

where $\bar{c}$ is defined in the problem.

2. Firm i maximizes

$$p_i\left[\bar{c} - \frac{1}{b}\log(p_i)\right]$$

The first order condition is

$$\bar{c} - \frac{1}{b}\log(p_i) - \frac{1}{b} = 0$$

and so the firm will charge $p_i = e^{b\bar{c}-1}$.

3. Using the normalization, profits are given by $\bar{c}$, and due to free entry profits must go to workers. A firm employs $\bar{l}$ units of labor, so $w = \frac{\bar{c}}{\bar{l}}$. The price normalization also implies $\bar{c} = \frac{1}{b}$, so we get $w = \frac{1}{b\bar{l}}$. From labor market clearing $N\frac{\bar{l}}{L} = 1$ we get the number of goods $N = \frac{L}{\bar{l}}$. Income of a worker is given by $wL = \frac{L}{b\bar{l}}$ and falls with $\bar{l}$ as profits have to be distributed among more workers.

4. The utility of a worker is

$$N\frac{1-e^{-b\bar{c}}}{b} = \frac{\bar{l}}{L}\frac{1-\frac{1}{e}}{b}$$

Only the ratio $\lambda \equiv \frac{\bar{l}}{L}$ matters, and higher productivity is associated with higher utility.

5. Managers receive what is left of profits after workers have been paid:

$$\omega(q) = \frac{1}{b} - w\frac{\bar{l}}{q}$$

6. The left hand side is the supply of workers, the right hand side is the demand for workers. Again only the ratio $\lambda = \frac{\bar{l}}{L}$ matters, and differentiating this yields

$$\frac{\partial q^*}{\partial \lambda} = \frac{\int_{q^*}^{q_{\max}} \frac{f(q)}{q} dq}{f(q^*)\left[1 + \frac{\lambda}{q^*}\right]}$$

Using the equation implicitly defining q^* one can also write

$$\frac{\partial q^*}{\partial \lambda}\frac{\lambda}{q^*} = \frac{F(q^*)}{f(q^*)\left[q^* + \lambda\right]}$$

Substituting the assumed distribution function and density yields

$$\frac{\partial q^*}{\partial \lambda}\frac{\lambda}{q^*} = \frac{q^* - q_{\min}}{q^* + \lambda} < 1,$$

which insures that $\frac{\bar{l}}{q^*}$ is increasing in $\bar{l}$.

7. The manager with ability q^* must be indifferent between managing and working, so we must have

$$wL = \frac{1}{b} - w\frac{\bar{l}}{q^*}$$

and so

$$w = \frac{\frac{1}{b}}{L + \frac{\bar{l}}{q^*}}$$

We can write

$$wL = \frac{\frac{1}{b}}{1 + \frac{\lambda}{q^*}}$$

8. The absolute wage level clearly falls. The number of firms and thus the number of managers is $N = 1 - F(q^*)$ and falls. If income is R, then utility is

$$U(R, N) = N\frac{1 - e^{-b\frac{R}{N}}}{b}$$

We already know that the income of a worker wL falls, which reduces utility. We also know that N falls. It remains to show that the fall in N also reduces utility. We have

$$\frac{\partial U}{\partial N} = \frac{1}{b}g\left(b\frac{R}{N}\right)$$

where

$$g(x) = 1 - e^{-x} - xe^{-x}.$$

We have $g(0) = 0$ and $g'(x) = 1 + xe^{-x}$, so utility is increasing in N.
We have

$$\frac{\omega(q)}{wL} = \frac{1}{bwL} - \frac{\lambda}{q} = 1 + \frac{\lambda}{q^*} - \frac{\lambda}{q}$$

Define

$$h(\lambda, q) = 1 + \lambda \left(\frac{1}{q^*(\lambda)} - \frac{1}{q} \right).$$

We have

$$\frac{\partial h(\lambda, q)}{\partial \lambda} = \frac{1}{q^*(\lambda)} - \frac{1}{q} - \lambda \frac{1}{(q^*)^2} \frac{\partial q^*}{\partial \lambda}$$

Evaluating this at $q = q^*$ yields

$$\frac{\partial h(\lambda, q^*)}{\partial \lambda} = -\lambda \frac{1}{(q^*)^2} \frac{\partial q^*}{\partial \lambda} < 0,$$

so inequality between workers and low-quality managers falls. Now

$$\frac{\partial^2 h(\lambda, q)}{\partial \lambda \partial q} = \frac{1}{q^2}$$

and

$$\lim_{q \to \infty} \frac{\partial h(\lambda, q)}{\partial \lambda} = \frac{1}{q^*(\lambda)} \left[1 - \frac{\partial q^*}{\partial \lambda} \frac{\lambda}{(q^*} \right] > 0,$$

so inequality between production workers and high-quality managers increases (although q_{max} may not be high enough to have an increase in inequality).
Finally we have

$$\frac{\omega(q')}{\omega(q)} = \frac{h(q', \lambda)}{h(q, \lambda)}$$

and this will be increasing if the elasticity

$$\frac{\partial h}{\partial \lambda} \frac{\lambda}{h} = \frac{\lambda \left(\frac{1}{q^*(\lambda)} - \frac{1}{q} \right) - \lambda^2 \frac{1}{(q^*)^2} \frac{\partial q^*}{\partial \lambda}}{1 + \lambda \left(\frac{1}{q^*(\lambda)} - \frac{1}{q} \right)}$$

is increasing in q. This is the case if

$$\frac{\lambda}{q^2} \left[1 + \lambda \left(\frac{1}{q^*(\lambda)} - \frac{1}{q} \right) \right] - \frac{\lambda}{q^2} \left[\lambda \left(\frac{1}{q^*(\lambda)} - \frac{1}{q} \right) - \lambda^2 \frac{1}{(q^*)^2} \frac{\partial q^*}{\partial \lambda} \right] > 0$$

which is satisfied.