# 1 Asset Prices: overview

- Euler equation
- C-CAPM
- equity premium puzzle and risk free rate puzzles
- Law of One Price / No Arbitrage
- Hansen-Jagannathan bounds
- resolutions of equity premium puzzle

# 2 Euler equation

- agent problem

$$\max \sum_{j=0}^{\infty} \sum_{s^t} \beta^t u\left(c_t\left(s^t\right)\right) \Pr\left(s^t\right)$$

$$\begin{aligned} c_t\left(s^t\right) + q_t^a\left(s^t\right) \cdot a_{t+1}\left(s^t\right) &\leq W_t\left(s^t\right) \\ W_{t+1}\left(s^{t+1}\right) &= y_{t+1}\left(s^{t+1}\right) + \left(q_{t+1}^a\left(s^{t+1}\right) + d_{t+1}\left(s^{t+1}\right)\right) a_{t+1}\left(s^t\right) \end{aligned}$$

- comment: $a_t$ and $q_t^a$ are vectors of length equal to the number of assets
- Euler equation

$$u'\left(c_t\right) q_t^{ai} = \beta E_t\left[u'\left(c_{t+1}\right)\left(q_{t+1}^{ai} + d_{t+1}^i\right)\right] \tag{1}$$

$$u'\left(c_t\right) = \beta E_t\left[u'\left(c_{t+1}\right) R_{t+1}^i\right]$$

$$1 = E_t\left[\beta \frac{u'\left(c_{t+1}\right)}{u'\left(c_t\right)} R_{t+1}^i\right] \tag{2}$$

- transversality condition

$$\lim_{j \to \infty} \beta^j E_0\left[u'\left(c_{t+j}\right) q_{t+j}^a a_{t+j}\right] = 0$$

- pricing formula

  repeated substitution of (1)

$$q_t^a = \sum_{j=1}^{\infty} \beta^j E_t\left[\frac{u'\left(c_{t+j}\right)}{u'\left(c_t\right)} d_{t+j}\right] \tag{3}$$

- no bubbles

- transversality and $s_t = 1$

- complete markets consistency check

  review A-D price with complete markets

$$q_{t+j}^{t}\left(s^{t}, s^{j}\right)=\beta \frac{u^{\prime}\left(c_{t+1}^{i}\left(s^{t}, s^{j}\right)\right)}{u^{\prime}\left(c_{t}^{i}\left(s^{t}\right)\right)} \operatorname{Pr}\left(s^{j} \mid s^{t}\right)$$

  $\rightarrow$ (3)

# 3 CCAPM (Consumption Capital Asset Pricing Model)

- make (2) and (3) operational:

  CCAPM $\equiv$ use aggregate consumption in above equations

- justifications:
  - equilibrium of representative agent economy (Lucas / Breeden)
  - equilibrium with complete markets (Constantinides)

    complete markets $\Longleftrightarrow$ Pareto Optima $\Longleftrightarrow$ representative consumer (weighted utility)

- back to Euler equation

$$1=E_{t}\left[\beta \frac{u^{\prime}\left(c_{t+1}\right)}{u^{\prime}\left(c_{t}\right)} R_{t+1}^{i}\right]$$

- Absence of arbitrage implies that there exists some $m_{t+1}$ such that

$$1=E_{t}\left[m_{t+1} R_{t+1}^{i}\right]$$

  THE empirically testable condition (again)

- intuitive decomposition

$$1=\beta E_{t}\left(\frac{u^{\prime}\left(c_{t+1}\right)}{u^{\prime}\left(c_{t}\right)}\right) E_{t}\left(R_{t+1}^{i}\right)+\beta cov_{t}\left(\frac{u^{\prime}\left(c_{t+1}\right)}{u^{\prime}\left(c_{t}\right)}, R_{t+1}^{i}\right)$$

  $\rightarrow$its the covariance that matters!

# 4 Equity Premium Puzzle

- Euler equations with data on $R^{\text{stock market}}$ and $R^{\text{bonds}}$

- simple log-normal calculation
- preferences and consumption

$$u'(c) = c^{-\gamma}$$

$$\begin{aligned}
\frac{c_{t+1}}{c_t} &= \bar{c}_\Delta \exp\left\{\varepsilon_c - \frac{1}{2}\sigma_c^2\right\} \\
\varepsilon_c &\backsim N\left(\mu_c, \sigma_c^2\right) \\
&\Rightarrow E\left(\frac{c_{t+1}}{c_t}\right) = \mu^c
\end{aligned}$$

- returns

$$\begin{aligned}
R^i &= \left(1+\bar{r}^i\right) \exp\left\{\varepsilon_i - \frac{1}{2}\sigma_i^2\right\} \\
\varepsilon_i &\backsim N\left(\mu_c, \sigma_c^2\right) \\
&\Rightarrow E\left(R^i\right) = R^i = 1+\bar{r}^i
\end{aligned}$$

- Euler

$$\begin{aligned}
1 &= \beta E\left[R^i \left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}\right] \\
1 &= \beta\left(1+\bar{r}^i\right)\left(\bar{c}_\Delta\right)^{-\gamma} E_t \exp\left(\varepsilon_i - \frac{1}{2}\sigma_i^2 - \gamma\varepsilon_c + \gamma\frac{1}{2}\sigma_c^2\right) \\
1 &= \beta\left(1+\bar{r}^i\right)\left(\bar{c}_\Delta\right)^{-\gamma} E_t \exp\left(\left(1+\gamma\right)\gamma\frac{1}{2}\sigma_c^2 - \gamma\sigma_{ic}\right)
\end{aligned}$$

- taking logs...

$$\log\left(1+\bar{r}^i\right) = -\log\beta + \gamma\log\bar{c}_\Delta - \left(1+\gamma\right)\gamma\frac{1}{2}\sigma_c^2 + \gamma\sigma_{ic}$$

- stocks and bonds:

$$\begin{aligned}
\bar{r}^f &\approx \log\left(1+\bar{r}^f\right) = -\log\beta + \gamma\log\bar{c}_\Delta - \left(1+\gamma\right)\gamma\frac{1}{2}\sigma_c^2 \quad (4)\\
\bar{r}^s &\approx \log\left(1+\bar{r}^s\right) = -\log\beta + \gamma\log\bar{c}_\Delta - \left(1+\gamma\right)\gamma\frac{1}{2}\sigma_c^2 + \gamma\sigma_{sc} \quad (5)
\end{aligned}$$

- premium:

$$\bar{r}^s - \bar{r}^f \approx \log\left(1+\bar{r}^s\right) - \log\left(1+\bar{r}^f\right) = \gamma\sigma_{sc} \quad (6)$$

Table removed due to copyright restrictions.

Kocherlakota, Narayana R. "The Equity Premium Puzzle: It's Still a Puzzle." *Journal of Economic Literature* 34, no. 1 (1996): 47 (Table 1).

- US data (from Mehra and Prescott):

$$\begin{aligned} \bar{r}^s &= 7\% \\ \bar{r}^f &= 1\% \\ \sigma_{rc} &= .219\% \end{aligned}$$

- Kocherlakota
- need $\gamma = 27$ to match (6)

  equity premium puzzle
- to match (4) we need $\gamma$ very high or very low risk free rate puzzle

Tables removed due to copyright restrictions.

Kocherlakota, Narayana R. "The Equity Premium Puzzle: It's Still a Puzzle." *Journal of Economic Literature* 34, no. 1 (1996): 42-71. (Tables 2 and 3).

# 5 Discount Factors: LOP and NA

I follow Cochrane and Hansen (1992) closely – great paper to read

- two periods "now" and "then" ($t$ and $t+1$ if you prefer)
- $J$ "fundamental" assets:
  - $x^j$ payoff "then"
  - $q^j$ "now" price

  $\rightarrow$ stack into $x$ and $q$ (column) vectors
- payoff space for "then"
$$P \equiv \{p : p = c \cdot x \text{ for some } c \in \mathbb{R}\}$$
- pricing function $\pi(p) : P \rightarrow \mathbb{R}$

  then $\pi(x) = q$

- definition: Law of One Price (LOP) holds if the pricing function is linear

$$\pi(c \cdot x) = c \cdot \pi(x) = c \cdot q$$

$\Rightarrow c \cdot x = c' \cdot x$ then $c \cdot q = c' \cdot q$ [1]

- definition: discount factor $y \in P$

$$\pi(p) = E(yp)$$

- *Riesz representation Theorem*

  LOP $\Leftrightarrow \exists$ (stochastic) discount factor $y \in P$

- Let $\mathcal{Y}$ be the set of all discount factors

- note: $y$ may be negative

- example:

$$y^* = x' (Exx')^{-1} q$$

  note: if $Exx'$ is non-singular then remove assets from $x$ until it is!
  a non-singular $Exx'$ means that (a) there is a risk-free asset (b) there are two ways of getting the same payoff

- Definition: No Arbitrage (NA) holds

$$\begin{aligned} p &\geq 0 \Rightarrow \pi(p) \geq 0 \\ p &> 0 \text{ (with positive prob.)} \Rightarrow \pi(p) > 0 \end{aligned}$$

- result NA $\Leftrightarrow \exists$ strictly positive discount factor $y > 0$
  Let $\mathcal{Y}^{++}$ be the set of all discount factors that are positive

- examples

$$m = \frac{\beta^t u'(c_{then})}{u'(c_{now})}$$

[1] proof:

$$\begin{aligned} \pi(c \cdot x) &= \pi(c' \cdot x) \\ c\pi(x) &= c'\pi(x) \\ cq &= c'q \end{aligned}$$

# 6 Hansen-Jagannathan bounds

- all theories:

$$\begin{aligned} q &= E(mp) \\ m &= f(\text{data, parameters}) \end{aligned}$$

  (see Cochrane's book)

- note $p^i/q^i$ is rate of return

- H-J bounds:

  diagnostic tool for models of $m$

- special case:

  data on a single excess return relative to some baseline asset

$$r = p/q - p^0/q^0$$

  then $\pi(r) = 0$ so that

$$\begin{aligned} 0 &= Emr = EmEr + cov(m, r) \\ &= EmEr + \sigma_m \sigma_r corr(m, r) \end{aligned}$$

$$-1 \leq \frac{EmEr}{\sigma_m \sigma_r} = corr(m, r) \leq 1$$

$$\left| \frac{EmEr}{\sigma_m \sigma_r} \right| \leq 1$$

$$\frac{\sigma_r}{|Er|} \leq \frac{\sigma_m}{Em}$$

  intuition: need volatile $\sigma_m$

- note: $Em = 1/R^f$ if there is a risk free rate $R^f$

- lets generalize:

  for any random vector $x$ we can consider the population regression:

$$m = a + x'b + e$$

  which just defines $e$ uniquely as having $\mathbb{E}e = 0$ and $cov(x, e) = 0$

- by definition $cov(e, x) = 0$

$$\Rightarrow var(m) \geq var(x'b)$$

- idea compute $x'b$ and $var(x'b)$ to get lower bound

  $\rightarrow$ check whether theories for $y$ pass this test

$$\begin{aligned} b &= [cov(x,x)]^{-1} cov(x,y) \\ a &= Ey - Ex'b \end{aligned}$$

- How to compute $b$?

  idea: if $x = p$ then theory helps...

- assume $x = p$ note that

$$cov(x,y) = q - E(y)E(x)$$

  so:

$$b = [cov(x,x)]^{-1} [q - E(y)E(x)]$$

$$var\left(x' [cov(x,x)]^{-1} [q - E(y)E(x)]\right) = var(x) [var(x)]^{-2} E(y)^2 E(x)^2$$

- if we knew $E(y)$ we have a lower bound

  otherwise $\Rightarrow$ feasible region for pair $(E(y), var(y))$

- convenient

  - no need to recompute lower bound for each theory
  - helps see *where* the theory fails

- 3 cases:

  - risk-less return

    $\rightarrow E(y)$ pinned down and risky return
  - one excess-return $q = 0$

    Sharpe ratio and market price of risk (what we did before!)
  - general case$\rightarrow$very flexible, see CH paper

- figures 2.1: excess

- 2.2, 2.3, 2.4 from CH paper

# 7 Resolutions (?)

## 7.1 Exotic Preferences

- Risk Aversion vs. IES

  (Weil / Epstein-Zin)

- first-order risk aversion

  (Epstein-Zin)

- habit persistence *e.g.* $u(c_t - \alpha c_{t-t})$

  (Abel / Campbell-Cochrane)

- loss-aversion

## 7.2 Heterogenous Agent Incomplete Markets

- uninsured idiosyncratic risk

  (Mankiw / Constantinides-Duffie)

- borrowing constraints (Euler with inequality)

  (Luttmer / Heaton-Lucas)

- constrained optima with limited commitment

  (Alvarez-Jermann)

## 7.3 Knightian Uncertainty

- risk vs. uncertainty

- fear of not understanding returns / uncertainty over probability distribution / desire for robust decisions (Hansen and Sargent)

## 7.4 No risk premium!

- no risk premium to explain...

- historical returns on stocks were unexpected

  (McGratten-Prescott)

- bonds are money → low return

- stocks more risky than sample (low probability of a crash)

  (see Reitz, Cochrane, Weitzman and Barro)

# 8 Conclusions

Risk premium puzzle

- great example of interplay between theory and data
- no strong consensus on resolution yet

  many new ideas
- new models should explore
- revisit the welfare costs of BCs

  (Alvarez and Jermann)