14.452: Economic Growth
Problem Set 1

Due date: November 6, 2009 in Recitation.

Exercise 1: Consider a modified version of the continuous-time Solow growth model where the aggregate production function is

$$F(K, L, Z) = L^{\beta} K^{\alpha} Z^{1-\alpha-\beta},$$

where Z is land, available in fixed inelastic supply. Assume that $\alpha + \beta < 1$, capital depreciates at the rate δ, and there is an exogenous saving rate of s.

1. First suppose that there is no population growth. Find the steady-state capital-labor ratio and the steady-state output level. Prove that the steady state is unique and globally stable.
2. Now suppose that there is population growth at the rate n, that is, $\dot{L}/L = n$. What happens to the capital-labor ratio and output level as $t \to \infty$? What happens to returns to land and the wage rate as $t \to \infty$?
3. Would you expect the population growth rate n or the saving rate s to change over time in this economy? If so, how?

Exercise 2: Consider the discrete-time Solow growth model with constant population growth at the rate n, no technological change and depreciation rate of capital equal to δ. Assume that the saving rate is a function of the capital-labor ratio, thus given by $s(k)$.

1. Suppose that $f(k) = Ak$ and $s(k) = s_0 k^{-1} - 1$. Show that if $A + \delta - n = 2$, then for any $k(0) \in (0, As_0/(1+n))$, the economy immediately settles into an asymptotic cycle and continuously fluctuates between $k(0)$ and $As_0/(1+n) - k(0)$. [Suppose that $k(0)$ and the parameters are given such that $s(k) \in (0,1)$ for both $k = k(0)$ and $k = As_0/(1+n) - k(0)$].
2. Now consider more general continuous production function $f(k)$ and saving function $s(k)$, such that there exist $k_1, k_2 \in R_+$ with $k_1 \neq k_2$ and

$$\begin{aligned} k_2 &= \frac{s(k_1) f(k_1) + (1-\delta) k_1}{1+n} \\ k_1 &= \frac{s(k_2) f(k_2) + (1-\delta) k_2}{1+n}. \end{aligned}$$

Show that when such (k_1, k_2) exist, there may also exist a stable steady state.

3. Prove that such cycles are not possible in the continuous-time Solow growth model for any (possibly non-neoclassical) continuous production function $f(k)$ and continuous $s(k)$.

4. What does the result in parts 1-3 imply for the approximations of discrete time by continuous time in the Solow model (suggested in Section 2.4 of the textbook)?

5. In light of your answer to part 4, what do you think of the cycles in parts 1 and 2?

6. Show that if $f(k)$ is nondecreasing in k and $s(k) = k$, cycles as in parts 1 and 2 are not possible in discrete-time either.

Exercise 3: Consider the Solow growth model with constant saving rate s and depreciation rate of capital equal to δ. Assume that population is constant and the aggregate output is given by the CES production function

$$F(A_K(t) K(t), A_L(t) L) = \left[\gamma (A_K(t) K(t))^{\frac{\sigma-1}{\sigma}} + (1-\gamma)(A_L(t) L)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}$$

where $\dot{A}_L(t)/A_L(t) = g_L > 0$ and $\dot{A}_K(t)/A_K(t) = g_K > 0$. Suppose the elasticity of substitution between capital and labor is less than one, $\sigma < 1$, and capital-augmenting technological progress is faster than labor-augmenting progess, $g_K \geq g_L$. Show that as $t \to \infty$, the economy converges to a BGP where the share of labor in national income is equal to 1, and capital, output and consumption all grow at the rate g_L. In light of this result, discuss the claims in the literature that capital-augmenting technological change is inconsistent with balanced growth. Why is the claim in the literature incorrect?

Exercise 4: Consider the Solow growth model in continuous time with constant saving rate s, depreciation rate δ, population growth rate n and labor-augmenting technological progress at rate $g > 0$, that is, suppose the production function function takes the form $Y(t) = F(K(t), A(t) L(t))$ where $\dot{A}(t)/A(t) = g > 0$. Recall that the effective capital-labor ratio $k(t) = K(t)/(A(t) L(t))$ is characterized by the differential equation

$$\frac{\dot{k}(t)}{k(t)} = \frac{sf(k(t))}{k(t)} - (\delta + g + n), \tag{1}$$

where $f(k(t)) = F(k(t), 1)$ is the production function normalized by effective labor, and the output per-capita is given by

$$y(t) = A(t) f(k(t)). \tag{2}$$

This exercises concerns an approximation for the growth rate of output per-capita around the steady-state.

1. Let $y^*(t) = A(t) f(k^*)$ denote the steady-state level of output per capita, that is, the level of output per capita that would apply if the effective capital-labor ratio were at its steady-state level and technology were at its time t level. Show that, in a neighborhood of the steady state, output per capita $y(t)$ can be approximated by the following differential equation:

$$\frac{\dot{y}(t)}{y(t)} \simeq g - (1 - \varepsilon_k(k^*))(\delta + g + n)(\log y(t) - \log y^*(t)), \qquad (3)$$

where $\varepsilon_k(k^*) = \frac{f'(k^*)k^*}{f(k^*)}$ is the elasticity of the production function $f(k)$ evaluated at k^*.

(Hint: First, consider the Taylor expansion of the right-hand side of Eq. (1) with respect to $\log(k(t))$ around $\log k^*$ and derive an approximation for $\dot{k}(t)/k(t)$. Second use this approximation along with Eq. (2) to derive an approximation for $\dot{y}(t)/y(t)$ as a function of the distance of effective capital-labor ratio from its steady-state, $\log k(t) - \log k^*$. Third, consider the Taylor expansion of Eq. (2) with respect to $\log(k(t))$ around $\log k^*$ to derive an approximation for the distance of output per-capita, $\log y(t) - \log y^*(t)$, in terms of the distance of effective capital-labor ratio, $\log k(t) - \log k^*$. Combine the last two steps to derive Eq. (3).)

2. Interpret Eq. (3). In particular, what does this equation imply about the growth rate of countries away from their steady-states? What happens to the growth rate as the countries approach their steady-states? Explain what determines the speed of convergence to the steady-state and provide an intuition.

14.452 Economic Growth
Fall 2009