14.384 Time Series Analysis, Fall 2008
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Supplementary to lectures given by Anna Mikusheva
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Recitation 1

Stationarity

Definition 1. *White noise* $\{e_t\}$ s.t. $Ee_t = 1$, $Ee_t e_s = 0$, $Ee_t^2 = \sigma^2$

Remark 2. $\{e_t\}$ can be white noise without being independent.

Definition 3. *strict stationarity* A process, $\{y_t\}$, is strictly stationarity if for each k, the distribution of $\{y_t, ..., y_{t+k}\}$ is the same for all t

Definition 4. *2nd order stationarity* $\{y_t\}$, is 2nd order stationary if Ey_t, Ey_t^2, and $\text{cov}(y_t, y_{t+k})$ do not depend on t

Remark 5. 2nd order stationarity is also called *covariance stationarity* or *weak stationarity*

Example 6. *ARCH*:
Let

$$\begin{aligned} y_t &= \sigma_t e_t \\ \sigma_t^2 &= \alpha + \theta y_{t-1}^2 \end{aligned}$$

with $e_t \sim iid(0, \sigma^2)$. This is an ARCH(1) process. It is covariance stationary. To show this, we first need to note that $E\sigma_t^2$ is finite

$$\begin{aligned} \sigma_t^2 &= \alpha + \theta(\sigma_{t-1}^2 e_{t-1}^2) \\ &= \alpha + \theta(\alpha + \theta\sigma_{t-2}^2 e_{t-2}^2)e_{t-1}^2 \\ &= \alpha \sum_{j=0}^{\infty} \theta^j (\prod_{k=1}^{j} e_{t-k}^2) \\ &\Rightarrow \\ E\sigma_t^2 &= \frac{\alpha}{1 - \theta\sigma^2} \end{aligned}$$

assuming that $\theta\sigma^2 \in [0, 1)$.
Now, we know that

$$E[y_t] = E[\sigma_t e_t] = 0$$

and

$$\begin{aligned} \text{cov}(y_t, y_{t+k}) &= E[\sigma_t e_t \sigma_{t+k} e_{t+k}] \\ &= \begin{cases} 0 & \text{if } k \neq 0 \\ \frac{\alpha\sigma^2}{1-\theta\sigma^2} & \text{if } k = 0 \end{cases} \end{aligned}$$

So this process is white noise.

ARMA

$ARMA(p,q)$:$a(L)y_t = b(L)e_t$, where a(L) is order p and $b(L)$ is order q, and $a(L)$ and $b(L)$ are relatively prime.

An ARMA representation is not unique. For example, an $AR(1)$ (with $|\rho| < 1$) is equal to an $MA(\infty)$, as we saw above. In fact, this is more generally true. Any $AR(p)$ with roots outside the unit circle has an MA representation.

Impluse response

Let $a(L)y_t = b(L)e_t$

Definition 7. *Impulse-response* of y_t is $\frac{\partial y_t}{\partial e_{t-i}}$

Definition 8. *cumulative effect of a shock* is $\sum_{i=0}^{\infty} \frac{\partial y_{t+i}}{\partial e_t}$

Example 9. *AR(1)* $(1-\rho L)y_t = e_t$

$$\frac{\partial y_t}{\partial e_{t-i}} = \rho^i$$
$$\sum_{i=0}^{\infty} \frac{\partial y_{t+i}}{\partial e_t} = \frac{1}{1-\rho}$$

Multivariate ARMA

Definition 10. *multivariate white noise* $e_t = \begin{bmatrix} e_{1t} \\ .. \\ e_{nt} \end{bmatrix}$, $E[e_t] = 0$, $E[e_t e_t'] = \Sigma$, $E[e_t e_s] = 0$

Definition 11. *multivariate ARMA(p,q)* $A(L)y_t = B(L)e_t$

We can manipulate multivariate ARMA (aka VARMA) representations just like we do univariate ones. A VARMA has an MA representation if all the roots of $A(L)$ are outside the unit circle. We'll cover this in more detail later.

Invertibility

[1] A lag polynomial, $A(L)$ is invertible, if given $B(L)$ and e_t, we can uniquely construct a stationary series y_t such that $A(L)y_t = B(L)e_t$. You have probably already heard that $A(L)$ is invertible if its roots are outside (or perhaps just not on the unit circle). To arrive at this fact, we first informally treat $A()$ and $B()$ as polynomials over $\mathbb{C}$, conjecture the result, and then verify that it is correct. As a function over $\mathbb{C}$, $\frac{B(z)}{A(z)}$ is well defined. We know that if $A()$ has no roots on or inside the unit circle, then $\frac{B(z)}{A(z)}$ is holomorphic on an open disc containing the unit circle.[2] This implies that $\frac{B(z)}{A(z)}$ has a power series representation on this disc:

$$\frac{B(z)}{A(z)} = \sum_{t=0}^{\infty} c_t z^t$$

[1] This is based on Chapter 7 of Van der Vaart's notes.

[2] If $A(z)$ simply has no roots on the unit circle, but has some inside, $\frac{B(z)}{A(z)}$ still has an absolutely convergent Laurent series on a ring containing the circle, $\frac{B(z)}{A(z)} = \sum_{t=-\infty}^{\infty} c_t z^t$. We could still call $A(L)$ invertible, but it's inverse now involves writing y_t as a combination of past and future e_t. This situation does not make much sense in econometrics, so we usually rule it out.

Importantly, this series is absolutely convergent on the disc, and on the unit circle in particular, so $\sum_{t=0}^{\infty} |c_t|$ is finite. This ensures that as long as $\sup_t Ee_t < \infty$, $\sum c_t e_t$ converges absolutely. It is straightforward to verify that $y_t = \sum c_t e_t$ satisfies $A(L)y_t = B(L)e_t$. It can also be shown that this is unique stationary solution to the ARMA equation. Finally, it is possible to show that if $A(L)$ has a root on the unit circle, then there is no stationary solution to $A(L)y_t = B(L)e_t$.

Role of stationarity

Stationarity plays is important for ensuring that an ARMA equation is well defined. Consider the following example (based on exercise 7.7 and theorem 7.8 of van der Vaart).

Example 12. Suppose $y_t = \rho y_{t-1} + e_t$. Let $\{e_t\}$ and rho be given. Pick any y_0. Then we can construct

$$y_t = \begin{cases} \rho y_{t-1} + e_t \, t > 0 \\ \frac{1}{\rho}(y_{t+1} - e_t) \, t < 0 \end{cases}$$

and these $\{y_t\}$ will satisfy the AR equation.

How can we reconcile this example with the $MA(\infty)$ representation, $y_0 = \sum_{i=0\infty} \rho^i e_{-i}$, which suggests that y_0 should be uniquely determined by $\{e_t\}$ and ρ?

The answer is that if $\{e_t\}$ is bounded, then the only y_0 that leads to bounded $\{y_t\}$ is $y_0 = \sum_{i=0\infty} \rho^i e_{-i}$. Or, to be more precise, if $\{e_t\}$ is covariance stationary, the only $\{y_t\}$ that satisfies the AR equation and is also covariance stationary is the one given by the $MA(\infty)$ equation.

Covariances

Definition 13. *auto-covariance* $\gamma_k \equiv \text{cov}(y_t, y_{t+k})$

Remark 14. In the next lecture, we will see that covariances are important for laws of large numbers and central limit theorem. Suppose y_t has auto-covariances γ_k. Consider the variance of the mean of T observations of y_t:

$$\begin{aligned} \text{Var}(\frac{1}{T}\sum_{t=1}^{T} y_t) =& \frac{1}{T^2}\left(\sum_{t=1}^{T}\text{Var}(y_t) + \sum_{t\neq s}\text{cov}(y_t, y_s)\right) \\ =& \frac{1}{T^2}\left(T\gamma_0 + 2(T-1)\gamma_1 + 2(T-2)\gamma_2 + ...\right) \\ =& \frac{1}{T}\left(\gamma_0 + 2\sum_{t=1}^{T}\gamma_t \frac{T-t}{T}\right) \end{aligned}$$

For a LLN we will need conditions on $\{\gamma_k\}$ to ensure that this series converges to zero. Similarly, the asympotic variance of $\sqrt{T}\bar{y}_T$ will depend on $\{\gamma_k\}$.

Definition 15. *auto-correlation* $\rho_k \equiv \frac{\gamma_k}{\gamma_0}$

Definition 16. *covariance generating function* $\gamma(\xi) = \sum_{i=-\infty}^{\infty} \gamma_i \xi^i$, where ξ is a complex number.

We can recover the auto-covariances from the covariance generating function by integrating it:

$$\gamma_k = \frac{1}{2\pi i}\int_C \gamma(z) z^{-k+1} dz \tag{1}$$

where C is a contour that goes counter-clockwise. If you're familar with complex analysis, you might recognize that (1) follows immediately from Cauchy's integral formula. The result can also be verified directly. The choice of contour does not matter here, so let's take the unit circle for convenience. That gives:

$$\begin{aligned}\frac{1}{2\pi i}\int_C \gamma(z)z^{-k}dz =&\frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_j \gamma_j e^{i\pi j}e^{-i\pi k}dz \\ =&\sum_j \gamma_j \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\pi(j-k)}dz \\ =&\gamma_k\end{aligned}$$

where the last line follows from the fact that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i\omega(n)}d\omega = \begin{cases}1 & \text{if } n = 0 \\ 0 & \text{if } n \neq 0\end{cases}$$

Remark 17. The covariance generating function is related to the spectral density, which we will cover in more detail later.

Definition 18. The *spectral density* of y is

$$s_Y(\omega) \equiv \frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\gamma_j e^{-i\omega j}$$

As with the covariance generating function, we can recover the auto-covariances of y from its spectral density.

Lemma 19. *Suppose y_t has absolute summable covariances ($\sum_{-\infty}^{\infty}|\gamma_j| < \infty$), then*

$$\gamma_k = \int_{-\pi}^{\pi} s_Y(\omega)e^{i\omega k}d\omega$$

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