

14.30 PROBLEM SET 6 SUGGESTED ANSWERS

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Problem 1

a. Let \tilde{X} denote the sum of the weight of the 100 sampled coins: $\tilde{X} = \sum_{i=1}^{100} X_i$. Now, \tilde{X} must be distributed normally, because it is a linear combination of independent normal random variables. Then $E(\tilde{X}) = E\left(\sum_{i=1}^{100} X_i\right) = \sum_{i=1}^{100} E(X_i) = 12800$, and $Var(\tilde{X}) = Var\left(\sum_{i=1}^{100} X_i\right) = \sum_{i=1}^{100} Var(X_i) = 100$, since the coins are assumed to be independent. Thus, if the Master is honest, $\tilde{X} \sim N(12800, 100)$.

We must choose some remedy R , such that we punish the Master whenever $\tilde{X} \leq 12800 - R$, and we want the probability that an honest Master is punished be no greater than 0.01. So we want to find R that satisfies

$$\begin{aligned} \Pr(\tilde{X} \leq 12800 - R) &= 0.01 \\ \Pr\left(\frac{\tilde{X} - 12800}{10} \leq -\frac{R}{10}\right) &= 0.01 \\ \Pr\left(Z \leq -\frac{R}{10}\right) &= 0.01 \end{aligned}$$

We can use our table to find that $-\frac{R}{10} = -2.33$, which implies $R = 23$.

b. Under the dishonest master, $\tilde{X} \sim N(12790, 100)$. For $R = 23$, the probability of catching this master is $\Pr(\tilde{X} \leq 12777) = \Pr\left(\frac{\tilde{X} - 12800}{10} \leq -\frac{12777 - 12800}{10}\right) = \Pr(Z \leq -1.3) \approx 0.097$. Thus, the chance that this dishonest Master eludes punishment is greater than 90%.

Problem 2

a. $S^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2$, so $E(S^2) = E\left(\frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2\right)$

$$\begin{aligned}
 E(S^2) &= E\left(\frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2\right) \\
 &= \frac{1}{n-1} E\left(\sum_{i=1}^n (X_i - \bar{X})^2\right) \\
 &= \frac{1}{n-1} E\left(\sum_{i=1}^n (X_i^2 - 2X_i\bar{X} + \bar{X}^2)\right) \\
 &= \frac{1}{n-1} E\left(\sum_{i=1}^n X_i^2 - 2\bar{X} \sum_{i=1}^n X_i + n\bar{X}^2\right) \\
 &= \frac{1}{n-1} E\left(\sum_{i=1}^n X_i^2 - 2n\bar{X}^2 + n\bar{X}^2\right) \\
 &= \frac{1}{n-1} E\left(\sum_{i=1}^n X_i^2 - n\bar{X}^2\right) \\
 &= \frac{1}{n-1} \left(\sum_{i=1}^n E(X_i^2) - nE(\bar{X}^2)\right) \\
 &= \frac{1}{n-1} (n\sigma^2 + n\mu^2 - n(\text{Var}(\bar{X}) + \mu^2)) \\
 &= \frac{1}{n-1} \left(n\sigma^2 - n\frac{\sigma^2}{n}\right) \\
 &= \frac{n-1}{n-1} (\sigma^2) \\
 &= \sigma^2
 \end{aligned}$$

b. To show $\bar{X} \xrightarrow{p} \mu$, we want to prove that for any $\varepsilon > 0$, as $n \rightarrow \infty$, $\Pr(|\bar{X} - \mu| < \varepsilon) \rightarrow 1$. We will denote the mean of a random sample of

size n by \bar{X}_n .

$$\begin{aligned}
 \Pr(|\bar{X}_n - \mu| < \varepsilon) &= \Pr(-\varepsilon < \bar{X}_n - \mu < \varepsilon) \\
 &= 1 - 2\Pr(\bar{X}_n - \mu > \varepsilon) \\
 &= 1 - 2\Pr\left(\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} > \frac{\varepsilon}{\sigma/\sqrt{n}}\right) \\
 &= 1 - 2\Pr\left(Z > \sqrt{n}\frac{\varepsilon}{\sigma}\right) \\
 &= 1 - 2\left(1 - \Pr\left(Z < \sqrt{n}\frac{\varepsilon}{\sigma}\right)\right) \\
 &= -1 + 2\Pr\left(Z < \sqrt{n}\frac{\varepsilon}{\sigma}\right)
 \end{aligned}$$

Now, as $n \rightarrow \infty$, for any $\varepsilon > 1$, $\Phi\left(\sqrt{n}\frac{\varepsilon}{\sigma}\right) \rightarrow 1$, so $\Pr(|\bar{X}_n - \mu| < \varepsilon) \rightarrow -1 + 2 = 1$.

Problem 3

a. Expand the expression for the price of the stock:

$$S_t = S_0 + \sum_{i=1}^t X_i$$

Thus the change in the stock's price after 700 periods is

$$\Delta S = S_{700} - S_0 = \sum_{i=1}^{700} X_i$$

b. Define

$$\bar{X} = \frac{1}{700} \sum_{i=1}^{700} X_i$$

Since 700 is "large," the Central Limit Theorem implies that

$$\begin{aligned}
 \frac{\sqrt{700}(\bar{X} - \mu)}{\sigma} &\sim N(0, 1) \\
 \mu &= 0.39(1) + 0.20(0) + 0.41(-1) \\
 &= -0.02 \\
 E(X_i^2) &= 0.39(1)^2 + 0.20(0)^2 + 0.41(-1)^2 \\
 &= 0.80 \\
 \sigma^2 &= 0.80 - (-0.02)^2 \\
 &= 0.7996
 \end{aligned}$$

Therefore

$$\bar{X} \sim N\left(-0.02, \frac{0.7996}{700}\right)$$

c. The probability that the stock is up at least 10 over the first 700 periods can be calculated as follows:

$$\begin{aligned}
 \Pr\left(\sum_{i=1}^{700} X_i \geq 10\right) &= \Pr\left(\bar{X} \geq \frac{1}{70}\right) \\
 &= \Pr\left(\frac{\bar{X} + 0.02}{\sqrt{0.7996/700}} \geq \frac{\frac{1}{70} + 0.02}{\sqrt{0.7996/700}}\right) \\
 &= \Pr(Z \geq 1.01) \\
 &= 1 - \Pr(Z \leq 1.01) \\
 &\approx 0.156
 \end{aligned}$$

Problem 4

We can denote the weights of each of the hundred booklets as X_1, \dots, X_{100} . This is a random sample from a population with mean 1 and standard deviation 0.05. We want to know the probability that the sum of the X_i is greater than 100.4:

$$\Pr\left(\sum_{i=1}^{100} X_i \geq 100.4\right) = \Pr(\bar{X} \geq 1.004)$$

The sample mean has the properties such that

$$E(\bar{X}) = E(X_i) = 1$$

$$St. Dev.(\bar{X}) = \sqrt{Var(\bar{X})} = \sqrt{\frac{Var(X_i)}{100}} = \frac{St. Dev(X_i)}{10} = 0.005$$

Then, by the central limit theorem, we know that (approximately)

$$\bar{X} \sim N\left(1, (0.005)^2\right)$$

Therefore,

$$\begin{aligned}
 \Pr(\bar{X} \geq 1.004) &= \Pr\left(Z \geq \frac{1.004 - 1}{0.005}\right) \\
 &= \Pr(Z \geq 0.8) \\
 &\approx 0.2119
 \end{aligned}$$

where Z is a standard normal random variable.