# Auctions 1:

# Common auctions & Revenue equivalence & Optimal mechanisms

## 1 Notable features of auctions

- Ancient "market" mechanisms. Widespread in use. A lot of varieties.

- Simple and transparent games (mechanisms). Universal rules (does not depend on the object for sale), anonymous (all bidders are treated equally).

- Operate well in the incomplete information environments. Seller (and sometimes bidders as well) does not know how the others value the object.

- Optimality and efficiency in broad range of settings.

- Probably the most active area of research in economics.

## 2 Notation (Symmetric IPV)

Independent private values setting with symmetric risk-neutral buyers, no budget constraints.

- Single indivisible object for sale.
- $N$ potential buyers, indexed by $i$. $N$ commonly known to all bidders.
- $X_i$ — valuation of buyer $i$ — maximum willingness to pay for the object.
- $X_i \sim F[0, \omega]$ with continuous $f = F'$ and full support.
- $X_i$ is private value (signal); all $X_i$ are $iid$, which is common knowledge.

## 3 Common auctions

SEALED-BID Auctions.

- First price sealed-bid auction:

  Each bidder submits a bid $b_i \in \mathbb{R}$ (sealed, or unobserved by the others). The winner is the buyer with the highest bid, the winner pays her bid.

- Second price sealed-bid auction:

  As above, the winner pays second highest bid — highest of the bids of the others.

- $K$th price auction:

  The winner pays the $K$th highest price.

- All-pay auction:

  All bidders pay their bids.

OPEN (DYNAMIC) Auctions.

- Dutch auction:

  The price of the object starts at some high level, when no bidder is willing to pay for it. It is decreased until some bidder announces his willingness to buy. He obtains the object at this price.

  Note: Dutch and First-price auctions are equivalent in strong sense.

- English auction:

  The price of the object starts at zero and increases. Bidders start active — willing to buy the object at a price of zero. At a given price, each bidder is either willing to buy the object at that price (active) or not (inactive). While the price is increasing, bidders reduce(*) their demands. The auction stops when only one bidder remains active. She is the winner, pays the price at which the last of the others stopped bidding.

  Note: English auction is in a weak sense equivalent to the second-price auction.

# 4 First-price auction

Payoffs

$$\Pi_i = \begin{cases} x_i - b_i, & \text{if } b_i > \max_{j \neq i} b_j, \\ 0, & \text{otherwise.} \end{cases}$$

**Proposition:** Symmetric equilibrium strategies in a first-price auction are given by

$$\beta^{\mathrm{I}}(x) = E\left[Y_1 | Y_1 < x\right],$$

where $Y_1 = \max_{j \neq i}\{X_j\}$.

Proof: Easy to check that it is eq.strat., let us derive it.

Suppose every other bidder except $i$ follows strictly increasing (and differentiable) strategy $\beta(x)$.

Equilibrium trade-off: Gain from winning versus probability of winning.

Expected payoff from bidding $b$ when receiving $x_i$ is

$$G_{Y_1}(\beta^{-1}(b)) \times (x_i - b).$$

FOC:

$$\frac{g(\beta^{-1}(b))}{\beta'(\beta^{-1}(b))}(x - b) - G(\beta^{-1}(b)) = 0.$$

In symmetric equilibrium, $b(x) = \beta(x)$, so FOC $\Rightarrow$

$$\begin{aligned} G(x)\beta'(x) + g(x)\beta(x) &= xg(x), \\ \frac{d}{dx}(G(x)\beta(x)) &= xg(x), \\ \beta(x) &= \frac{1}{G(x)}\int_0^x yg(y)dy, \\ &= E\left[Y_1 | Y_1 < x\right]. \end{aligned}$$

In the first price auction expected payment is

$$\begin{aligned} m^{\mathrm{I}}(x) &= \Pr[\text{Win}] \times b(x) \\ &= G(x) \times E\left[Y_1 | Y_1 < x\right]. \end{aligned}$$

# 5 Examples:

1. Suppose values are uniformly distributed on $[0, 1]$.

   $F(x) = x$, then $G(x) = x^{N-1}$ and

   $$\beta^{\mathsf{I}}(x) = \frac{N-1}{N}x.$$

2. Suppose values are exponentially distributed on $[0, \infty)$.

   $F(x) = 1 - e^{-\lambda x}$, for some $\lambda > 0$ and $N = 2$, then

   $$\begin{aligned} \beta^{\mathsf{I}}(x) &= x - \int_0^x \frac{F(y)}{F(x)} dy \\ &= \frac{1}{\lambda} - \frac{xe^{-\lambda x}}{1 - e^{-\lambda x}}. \end{aligned}$$

   Note that if, say for $\lambda = 2$, $x$ is very large the bid would not exceed 50 cents.

# 6 Second-price auction

**Proposition:** In a second-price sealed-bid auction, it is a weakly dominant strategy to bid

$$\beta^{\mathsf{II}}(x) = x.$$

In the second price auction expected payment of the winner with value $x$ is the expected value of the second highest bid given $x$, which is the expectation of the second-highest value given $x$.

Thus, expected payment in the second-price auction is

$$\begin{aligned} m^{\mathsf{I}}(x) &= \Pr[\text{Win}] \times E\left[Y_1 | Y_1 < x\right] \\ &= G(x) \times E\left[Y_1 | Y_1 < x\right]. \end{aligned}$$

# 7 Notation (IPV)

Independent private values setting with risk-neutral buyers, no budget constraints. Not necessarily symmetric.

- Single indivisible object for sale.
- $N$ potential buyers, indexed by $i$. $N$ commonly known to all bidders.
- $X_i$ — private valuation of buyer $i$ — maximum willingness to pay for the object.
- $X_i \sim F_i[0, \omega_i]$ with continuous $f_i = F_i'$ and full support, independent across buyers.
- $\mathcal{X} = \times_{i=1}^{N} \mathcal{X}_i$, $\mathcal{X}_{-i} = \times_{j \neq i} \mathcal{X}_j$, $f(\mathbf{x})$ is joint density.

# 8 Mechanisms

A selling mechanism $(\mathcal{B}, \pi, \mu)$ :

- $\mathcal{B}_i$ — a set of messages (or bids) for player $i$.
- $\pi : \mathcal{B} \to \Delta$ — allocation rule; here $\Delta$ is the set of probability distributions over $N$.
- $\mu : \mathcal{B} \to \mathbb{R}^n$ — payment rule.

Example: First- and second-price auctions.

Every mechanism defines an incomplete information game:

- $\beta_i : [0, \omega_i] \to \mathcal{B}_i$ is a strategy;
- Equilibrium is defined accordingly.

# 9 Revelation principle

*Direct mechanism* $(\mathbf{Q}, \mathbf{M})$:

- $\mathcal{B}_i = \mathcal{X}_i$;
- $\mathbf{Q} : \mathcal{X} \to \Delta$, where $Q_i(\mathbf{x})$ is the probability that $i$ gets the object.
- $\mathbf{M} : \mathcal{X} \to \mathbb{R}^n$, where $M_i(\mathbf{x})$ is the expected payment by $i$.

**Proposition: (Revelation principle)** Given a mechanism and an equilibrium for that mechanism, there exist a direct mechanism in which:

1. it is an equilibrium for each buyer to report truthfully, and
2. the outcomes are the same.

Proof: Define $\mathbf{Q}(\mathbf{x}) = \pi(\beta(\mathbf{x}))$ and $\mathbf{M}(\mathbf{x}) = \mu(\beta(\mathbf{x}))$. Verify.

# 10 Incentive compatibility

Define $q_i(z_i)$ and $m_i(z_i)$ to be a probability that $i$ gets the object and her expected payment from reporting $z_i$ while every other bidder reports truthfully:

$$\begin{aligned} q_i(z_i) &= \int_{\mathcal{X}_{-i}} Q_i(z_i, \mathbf{x}_{-i}) f_{-i}(\mathbf{x}_{-i}) \mathbf{dx}_{-i}, \\ m_i(z_i) &= \int_{\mathcal{X}_{-i}} M_i(z_i, \mathbf{x}_{-i}) f_{-i}(\mathbf{x}_{-i}) \mathbf{dx}_{-i}. \end{aligned}$$

Expected payoff of the buyer $i$ with value $x_i$ and reporting $z_i$ is

$$q_i(z_i)x_i - m_i(z_i).$$

Direct mechanism $(\mathbf{Q}, \mathbf{M})$ is *incentive compatible (IC)* if $\forall i, x_i, z_i$, *equilibrium payoff function* $U_i(x_i)$ satisfies

$$U_i(x_i) \equiv q_i(x_i)x_i - m_i(x_i) \geq q_i(z_i)x_i - m_i(z_i).$$

$IC$ implies that

$$U_i(x_i) = \max_{z_i \in \mathcal{X}_i} \{q_i(z_i)x_i - m_i(z_i)\}$$

— maximum of a family of affine functions, thus, $U_i(x_i)$ is convex.

By comparing expected payoffs of buyer $i$ with $z_i$ of reporting truthfully ($z_i$) and of reporting $x_i$, we obtain:

$$U_i(z_i) \geq U_i(x_i) + q_i(x_i)(z_i - x_i),$$

so $q_i(x_i)$ is the slope of the line that "supports" $U_i(x)$ at $x_i$.

$U_i$ convex $\longrightarrow$

$U_i$ is absolutely continuous $\longrightarrow$

$U_i$ is differentiable almost everywhere ($U_i'(x_i) = q_i(x_i)$ and so $q_i(x_i)$ is non-decreasing) $\longrightarrow$

$U_i$ is the integral of its derivative:

$$U_i(x_i) = U_i(0) + \int_0^{x_i} q_i(t_i)dt_i.$$

**Conclusion:** The expected payoff to a buyer in an incentive compatible direct mechanism $(\mathbf{Q}, \mathbf{M})$ depends (up to a constant) *only* on the allocation rule $\mathbf{Q}$.

Note: $IC \iff q_i(x)$ is non-decreasing.

# 11 Revenue Equivalence

**Proposition: (Revenue Equivalence)** If the direct mechanism $(\mathbf{Q}, \mathbf{M})$ is incentive compatible, then $\forall i, x_i$ the expected payment is

$$m_i(x_i) = m_i(0) + q_i(x_i)x_i - \int_0^{x_i} q_i(t_i)dt_i.$$

Thus, the expected payments (and so the expected revenue to the seller) in any two IC mechanism with the same allocation rule are equivalent up to a constant.

**Proof:** $U_i(x_i) = q_i(x_i)x_i - m_i(x_i)$, $U_i(0) = -m_i(0)$. Substitute.

## 11.1 An application of Revenue Equivalence

Consider symmetric (iid) environment.

In the second-price auction

$$\beta^{\mathsf{II}}(x) = x.$$

and

$$m^{\mathsf{II}}(x) = G(x) \times E\left[Y_1 | Y_1 < x\right].$$

In the first-price auction, since

$$m^{\mathsf{I}}(x) = G(x) \times b(x)$$

we obtain

$$\beta^{\mathsf{I}}(x) = E\left[Y_1 | Y_1 < x\right]$$

In the all-pay auction

$$m^{\mathsf{A}}(x) = \beta^{\mathsf{A}}(x) = G(x) \times E\left[Y_1 | Y_1 < x\right].$$

# 12 Individual rationality

Direct mechanism $(\mathbf{Q}, \mathbf{M})$ is *individually rational (IR)* if $\forall i, x_i$,

$$U_i(x_i) \geq 0.$$

**Corollary:** If mechanism $(\mathbf{Q}, \mathbf{M})$ is $IC$ then it is $IR$ if for all buyers $U_i(0) \geq 0$ (or $m_i(0) \leq 0$).

# 13 Optimal mechanisms

Consider direct mechanism $(\mathbf{Q}, \mathbf{M})$.

The expected revenue to the seller is

$$\begin{aligned} E[R] &= \sum_{i \in \mathcal{N}} E[m_i(X_i)], \text{ where} \\ E[m_i(X_i)] &= \int_0^{\omega_i} m_i(x_i) f_i(x_i) dx_i \\ &= m_i(0) + \int_0^{\omega_i} q_i(x_i) x_i f_i(x_i) dx_i \\ &\quad - \int_0^{\omega_i} \int_0^{x_i} q_i(t_i) f_i(x_i) dt_i dx_i. \end{aligned}$$

The last term is equal to (with changing variables of integration)

$$\int_0^{\omega_i} \int_{t_i}^{\omega_i} q_i(t_i) f_i(x_i) dx_i dt_i = \int_0^{\omega_i} (1 - F_i(t_i))\, q_i(t_i) dt_i.$$

Substituting back

$$
\begin{aligned}
& E[m_i(X_i)] \\
= \ & m_i(0) + \int_0^{\omega_i} \left( x_i - \frac{1 - F_i(x_i)}{f_i(x_i)} \right) q_i(x_i) f_i(x_i) dx_i \\
= \ & m_i(0) + \int_{\mathcal{X}} \left( x_i - \frac{1 - F_i(x_i)}{f_i(x_i)} \right) Q_i(\mathbf{x}) f(\mathbf{x}) \mathbf{dx}.
\end{aligned}
$$

*Optimal mechanism* maximizes $E[R]$ subject to: $IC$ and $IR$.

# 14 Solution

Define the *virtual valuation* of a buyer with value $x_i$ as

$$\psi_i(x_i) = x_i - \frac{1 - F_i(x_i)}{f_i(x_i)}.$$

Then seller should choose $(\mathbf{Q}, \mathbf{M})$ to maximize

$$\sum_{i \in \mathcal{N}} m_i(0) + \int_{\mathcal{X}} \left( \sum_{i \in \mathcal{N}} \psi_i(x_i) Q_i(\mathbf{x}) \right) f(\mathbf{x}) \mathbf{dx}.$$

Look at $\sum_{i \in \mathcal{N}} \psi_i(x_i) Q_i(\mathbf{x})$. It is best to give the highest weights $Q_i(\mathbf{x})$ to the maximal $\psi_i(x_i)$.

Design problem is regular if for $\forall i$, $\psi_i(\cdot)$ is an increasing function of $x_i$.

Regularity would imply incentive compatibility of the optimal mechanism.

The following is the optimal mechanism $(\mathbf{Q}, \mathbf{M})$:

- Allocation rule $\mathbf{Q}$:

$$Q_i(\mathbf{x}) > 0 \iff \psi_i(x_i) = \max_{j \in \mathcal{N}} \psi_j(x_j) \geq 0.$$

  ($q_i(x_i)$ is non-decreasing if $\psi_i(x_i)$ is, so we have $IC$.)

- Payment rule $\mathbf{M}$: (implied by $IC$ and $IR$)

$$M_i(\mathbf{x}) = Q_i(\mathbf{x}) x_i - \int_0^{x_i} Q_i(z_i, \mathbf{x}_{-i}) dz_i.$$

  ($M_i(0, \mathbf{x}_{-i}) = 0$ for all $\mathbf{x}_{-i}$ and so $m_i(0) = 0$, so we have $IR$.)

Define

$$y_i(\mathbf{x}_{-i}) = \left\{ \text{inf } z_i : \psi_i(z_i) \geq 0 \text{ and } \psi_i(z_i) \geq \max_{j \neq i} \psi_j(x_j) \right\}$$

— the smallest value for $i$ that "wins" against $\mathbf{x}_{-i}$.

Thus,

$$Q_i(z_i, \mathbf{x}_{-i}) = \begin{cases} 1, & \text{if } z_i > y_i(\mathbf{x}_{-i}), \\ 0, & \text{if } z_i < y_i(\mathbf{x}_{-i}). \end{cases}$$

We have

$$\int_0^{x_i} Q_i(z_i, \mathbf{x}_{-i}) = \begin{cases} x_i - y_i(\mathbf{x}_{-i}), & \text{if } z_i > y_i(\mathbf{x}_{-i}), \\ 0 &, \text{ if } z_i < y_i(\mathbf{x}_{-i}). \end{cases}$$

and, so,

$$M_i(\mathbf{x}) = \begin{cases} y_i(\mathbf{x}_{-i}), & \text{if } Q_i(\mathbf{x}) = 1, \\ 0 &, \text{ if } Q_i(\mathbf{x}) = 0. \end{cases}$$

**Proposition:** Suppose the design problem is regular and symmetric. Then a second-price auction with a reserve price $r^* = \psi^{-1}(0)$ is an optimal mechanism.