Class notes on SLP's Section 4.1
The Principle of Optimality

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Here are some results that are meant to complement Stokey and Lucas with Prescott's (SLP) treatment of the Principle of Optimality.

Theorems 4.4 and 4.5 are modified without weakening their applicability so that they are exact converses of each other. The same is done for Theorems 4.2 and 4.3 for the case where the supremums are attained and the value functions are finite valued – which are the situation one usually wishes to work with in any case.

An example is displayed for Theorem 4.3 that is meant to dispel the conjecture that the need for some boundedness condition is only due to "pathological" cases where $V^* = \infty$.

1 Theorem 4.4. and 4.5 as Exact Converses

Suppose $|V^*(x_0)| < \infty$ and $x^* \in \Pi(x_0)$ is optimal:

$$V^*(x_0) = u(x^*)$$

Then,

$$V^*(x_t^*) = F\left(x_t^*, x_{t+1}^*\right) + \beta V^*\left(x_{t+1}^*\right),$$

by virtue of Theorem 4.3. Thus:

$$\begin{aligned} V^*(x_0^*) &= \sum_{t=0}^{n} \beta^t F\left(x_t^*, x_{t+1}^*\right) + \beta^{n+1} V^*\left(x_{n+1}^*\right) \\ &= u_n(x^*) + \beta^{n+1} V^*\left(x_{n+1}^*\right) \\ &= u(x^*) + \lim_{n\to\infty} \beta^{n+1} V^*\left(x_{n+1}^*\right) \end{aligned}$$

so that $\lim_{n\to\infty}\beta^n V^*(x_n^*) = 0$, implying $\limsup_{n\to\infty}\beta^n V^*(x_n^*) \le 0$ as in Theorem 4.5. Putting this result together together with Theorem 4.5 it is clear that the condition $\limsup_{n\to\infty}\beta^n V^*(x_n^*) \le 0$ in Theorem 4.5 is *never weaker* than $\lim_{n\to\infty}\beta^n V^*(x_n^*) = 0$, when $V^*(x_0)$ is finite.

Consequently, with the following modification Theorems 4.4 and 4.5 become exact converses of each other.

Theorem 4.4' and 4.5' *Let* X, Γ, F, *and* β *satisfy assumptions 4.1 and 4.2. Suppose* $V^*(x_0)$ is finite. *Then* $x^* \in \Pi(x_0)$ *attains the supremum for initial state* x_0 **if and only if**

$$V^*(x_t^*) = F\left(x_t^*, x_{t+1}^*\right) + \beta V^*\left(x_{t+1}^*\right) \tag{1}$$

and

$$\lim_{n\to\infty}\beta^n V^*(x_n^*) = 0$$

(or equivalently $\limsup_{n\to\infty}\beta^n V^*(x_n^*) = 0$).

If $V^*(x_0)$ is not finite then (1) continues to hold by Theorem 4.3 in SLP. If $V^*(x_0) = \infty$ then there is at least one x^* with $u(x^*) = \infty$. Since (1) holds it must be true that $V^*(x_t^*) = \infty$ for all $t \ge 0$ [since F is finite]. Thus Theorem 4.5 in SLP never helps locate a maximum in this case.

If $V^*(x_0) = -\infty$ then $u(x) = -\infty$ for all $x \in \Pi(x_0)$, so all sequences are optimal. In this case $V^*(x_t) = -\infty$ for all $t \ge 0$ [since F is finite]. Consequently, in this case we needed no help in finding an optimum.

It is clear that the cases $V^*(x_0) = \infty$ and $V^*(x_0) = -\infty$ must be treated differently from the finite case. However, the previous remarks show that our modifications do not reduce the applicability of Theorems 4.4 and 4.5.

2 Another Useful Variant to Theorem 4.3

Here is a weakening of the conditions for Theorem 4.3 for the case in which the supremum is attained in the FE. Note that this simple result is different, and thus complements, exercise 4.3. In particular, note that this result can be applied to all solutions bounded from below. Thus it applies to the supremum of the SLP's example immediately after Theorem 4.5 (whereas SLP's Theorem 4.3 does not apply).

Later we build on this result to show an even more interesting result that allows us to write versions of Theorems 4.2 and 4.3 as exact converses of each other.

Theorem 4.3'. *Let X, Γ, F, and β satisfy assumptions 4.1 and 4.2. Suppose V solves FE, that the supremum in FE is attained for all x,* and that $|V(x)| < \infty$ for all $x \in X$. *Then for each x_0 there is an $x^* \in \Pi(x_0)$ such that $V(x_t^*) = F(x_t^*, x_{t+1}^*) + \beta V(x_{t+1}^*)$ for $t = 0, 1, 2...$ If in addition, $V(x_0) = u(x^*)$, or equivalent $\lim_{t\to\infty} \beta^t V(x_t^*) = 0$ and*

$$\limsup \beta^t V(x_t) \geq 0 \tag{2}$$

for all $x \in \Pi(x_0)$, for all $x_0 \in X$, then $V^*(x) = V(x)$.

Proof. After repeated substitutions of the FE's inequality

$$V(x_t) \geq F(x_t, x_{t+1}) + \beta V(x_{t+1})$$

for all $x_{t+1} \in \Gamma(x_t)$, all $x_t \in X$, and taking lim sup on both sides yields

$$V(x_0) \geq u(x) + \limsup \beta^t V(x_t) \geq u(x)$$

for all feasible plans $x \in \Pi(x_0)$ (for the first inequality, we are using the property that $\limsup(x_n + y_n) = \lim x_n + \limsup y_n$ whenever $\lim x_n$ exists and is finite). Since $V(x_0) = u(x^*)$ for some x^*. It follows that $V = V^*$. ■

Remark 1. Note that in applications we will always want to be able to verify $\lim_{t\to\infty} \beta^t V(x_t^*) = 0$, so as to apply Theorem 4.5' once we have established that $V = V^*$. Thus the important additional condition above is that $\limsup \beta^t V(x_t) \geq 0$ *for all* $x \in \Pi(x_0)$, *all* $x_0 \in X$.

Remark 2. We now show an example that illustrates that the exact converse of this result is not true. There are V^* that do not satisfy $\limsup \beta^t V^*(x_t) \geq 0$ for all feasible plans. Let $X = \mathbb{R}_+$, $F(x, y) = -x$, $\Gamma(x) = \left\{0, \frac{x}{\beta}\right\}$ and any $\beta \in (0, 1)$. Then, $v^*(x) = -x$ and the supremum is attained by $x_{t+1} = 0$ for $t \geq 0$. This problem satisfies A1 and A2 as well. However, note that $\hat{x}_t = x_0 \beta^{-t}$ is feasible but that $\limsup \beta^t v(\hat{x}_t) = \lim \beta^t v(\hat{x}_t) = -x_0 < 0$ for $x_0 > 0$, thus violating condition (2). Thus, an exactly converse to our version of Theorem 4.3 is not possible (not all supremums will satisfy (2)). However, note that in this example $\hat{x}$ yields $u(\hat{x}) = -\infty$; in the last section we use this insight to produce a modification of Theorems 4.2 and 4.3 that are exact converses of each other.

3 Remark on SLP's Theorem 4.2 vs 4.3

Suppose V^* is finite and attained by x^*, i.e. $V^*(x) = u(x^*)$, then we know that $\lim_{t\to\infty} \beta^t V^*(x_t^*) = 0$. Thus the extra requirement in the various variants of Theorem 4.3 [as in exercise 4.3 of SLP or the result above] vs. the conclusion of Theorem 4.2 is that some related boundedness condition holds for *all other* plans, not just the optimal plan. This fact will be exploited below.

4 An Example for SLP's Theorem 4.3

Intentions. We will construct a dynamic problem with the property that V^* satisfies: (i) $V^*(x) < \infty$ for each $x \in X$, (ii) $V^*(x_0)$ is attained by some feasible plan (i.e. we can use max instead of sup), and (iii) the boundedness condition of Theorem 4.3, i.e. that

$$\lim_{t\to\infty} \beta^t V^*(x_t) = 0$$

for all feasible plans. For this same problem we'll show that the related FE has another solution, $V \neq V^*$, that does not satisfy $\lim_{t\to\infty} \beta^t V(x_t) = 0$ for all feasible plans.

Thus the intention is to produce an example for the need of the extra condition required for the converse to Theorem 4.2 (Theorem 4.3) that is "less pathological" in some senses to the one included in SLP. We do this by altering the example in SLP to satisfy the above three conditions.

The Primitives $[F, \Gamma, X, \beta]$. Let F be as in the example used in class, taken from SLP, i.e. $F(x, y) = x - \beta y$ and let $X = \mathbb{R}$. Now define Γ as follows,

$$\Gamma(x) = \begin{cases} \left\{-\frac{x}{\beta}\right\} & \text{if } x > 0 \\ \left\{\frac{x}{\beta}\right\} & \text{if } x \leq 0 \end{cases}$$

Supremum's Three Properties. There is only one feasible plan. Thus the supremum is obtained and $V^*(x) = 2x$ if $x > 0$ and $V^*(x) = 0$ if $x \leq 0$. Thus we have properties (i) and (ii).

As for property (iii) note that the only feasible sequence has $x_t \to -\infty$ so eventually $V^*(x_t) = 0$, thus $\lim_{t\to\infty} \beta^t V^*(x_t) = 0$ for all feasible plans, i.e. V^* satisfies the conditions of Theorem 4.3.

The other solution to the FE. Next note that the functional equation,

$$v(x) = \sup_{y \in \Gamma(x)} \{x - \beta y + \beta v(y)\},$$

always admits the solution $V(x) = x$ for all $x \in X$ [this is true for any non-empty Γ]. However this solution does *not* satisfy $\lim_{t\to\infty} \beta^t V^*(x_t) = 0$ for all feasible plans. To see this, consider the only feasible plan starting from positive x_0: for $x_0 > 0$, $\hat{x}_1 = -\hat{x}_0/\beta$ and $\hat{x}_{t+1} = \hat{x}_t/\beta$ for $t \geq 1$ (that is $\hat{x}_t = -\beta^{-t} x_0$) which yields $\lim_{t\to\infty} \beta^t V(\hat{x}_t) = -x_0$. Thus V does not satisfy the conditions of Theorem. 4.3 in SLP.

5 Theorems 4.2 and 4.3 as Exact Converses

This section resulted from a collaboration with Marek Pycia. We now unite Theorems 4.2 and 4.3.

Theorem 4.2' and 4.3'. *Let X, Γ, F, and β satisfy assumptions 4.1 and 4.2. A function $V : X \to \mathbb{R}$ (finite valued) attains the maximum in SP (that is $V = V^*$)* **if and only if:**

(i) *V solves FE*

(ii) *the maximum in FE is attained for all x_0 by some $x^* \in \Pi(x_0)$, that is,*

$$V(x_t^*) = F\left(x_t^*, x_{t+1}^*\right) + \beta V\left(x_{t+1}^*\right) \text{ for } t \geq 0.$$

In addition $V(x_0) = u(x^*)$, or equivalently,

$$\lim_{t\to\infty} \beta^t V(x_t^*) = 0.$$

(iii) *for all $x_0 \in X$ and $x \in \Pi(x_0)$ with $u(x) > -\infty$*

$$\limsup \beta^t V(x_t) \geq 0 \tag{3}$$

(or alternatively $\liminf \beta^t V(x_t) \geq 0$*)*

Proof. That a V satisfying (i)-(iii) is the supremum in SP was proved above in our variations and extensions of Theorems 4.3, 4.4 and 4.5.

As for the converse, suppose V^* attains the maximum in SP and is finite, we need to show that (i)-(iii) holds. Part (i) is implied by Theorem 4.2 in SLP. Part (ii) is implied by our version of Theorem 4.4. We need only prove (iii).

Take any $x_0 \in X$ and $x \in \Pi(x_0)$ such that $u(x) > -\infty$. Because $V^*(x) < \infty$ then $u(x) < \infty$. Also, since V^* is the supremum we have that

$$\beta^n V^*(x_n) \geq \sum_{t=n}^{\infty} \beta^t F(x_t, x_{t+1}) \tag{4}$$

since

$$u_n(x) = \sum_{t=0}^{n} \beta^t F(x_t, x_{t+1})$$

converges to a real number, (i.e. $-\infty < u(x) < \infty$) it follows that the "tail" must converge to zero:

$$\lim_{n\to\infty} \sum_{t=n}^{\infty} \beta^t F(x_t, x_{t+1}) = 0$$

The result then follows by taking the $\liminf$ or $\limsup$ in (4) using the property that $\limsup(x_n + y_n) = \lim x_n + \limsup y_n$ or $\liminf(x_n + y_n) = \lim x_n + \liminf y_n$, when $\lim x_n$ exists and is finite. ■

Remark. Exercise 4.3 in SLP examines the condition that

$$\limsup \beta^n V(x_n) \leq 0 \tag{5}$$

for all feasible plans $x \in \Pi(x_0)$, all x. This condition together with an additional condition are shown to establish $V = V^*$. The Theorem above shows that, in general, in such a case we will also obtain:

$$\liminf \beta^n V(x_n) \geq 0$$

all $x_0 \in X$ and $x \in \Pi(x_0)$ as long as $u(x) > -\infty$. Consequently,

$$\lim_{n\to\infty} \beta^n V(x_n) = 0$$

all $x_0 \in X$ and $x \in \Pi(x_0)$ and $u(x) > -\infty$.