14.123 Microeconomic Theory III. 2014

Problem Set 3. Solution.

Anton Tsoy

1. I work with certainty equivalent as in the lecture.

   1.1 Certainty equivalent of the individual's continuation utility is $(\mu - \frac{1}{2}\alpha\sigma^2)(T - t + 1) + X_{i1} + \cdots + X_{i(t-1)}$, and the individual is willing to sell the asset for price $P_{it} = (\mu - \frac{1}{2}\alpha\sigma^2)(T - t + 1) + X_{i1} + \cdots + X_{i(t-1)}$. Expected value of the asset to the company is $\mu(T - t + 1) + X_{i1} + \cdots + X_{i(t-1)} > P_{it}$ and so, the company should make such offer.

   1.2 The company buys all assets at date 1 paying $\Pi_1 = (\mu - \frac{1}{2}\alpha\sigma^2)Tn$ and the return is $\sum_i Y_i$. Certainty equivalent of the dividend of the new individual equals $\mu T - \frac{T}{2n}\alpha\sigma^2 - \frac{\Pi_1}{n} = \frac{1}{2}\alpha\sigma^2 T(1 - \frac{1}{n})$, which is the maximal price that a new individual is willing to pay for a $1/n$ share.

   1.3 Suppose that company could buy the assets only starting from date $t = 2$. Then it pays for the assets price $\Pi_2 = (\mu - \frac{1}{2}\alpha\sigma^2)(T - 1)n + \sum_i X_{i1}$. Certainty equivalent of the dividend equals $\frac{1}{2}\alpha\sigma^2(T - 1)(1 - \frac{1}{n})$.

   1.4 A new individual is willing to pay less for the share of the company with restricted trades, since he expects that the assets will be purchased by the company at $t = 2$ at a higher price.

2. I start by setting up the general problem. The Bellman equation for the problem is

$$V(\omega) = \max_{s\in[0,\omega]} u(\omega - s) + \delta(1 - \rho)\mathbb{E}[V(sr)].$$

The first-order condition for this problem together with the envelope condition $V'(\omega) = u'(\omega - s)$ gives

$$u'(\omega - s) = \delta(1 - p)\mathbb{E}[ru'(rs - s_+)], \tag{1}$$

where s_+ is the savings in the next round.

   2.1 In (1), set $r = 1$, $(c_0^*)^{-\rho} = \delta(1 - p)(\omega - c_0^*)^{-\rho}$ or $c_0^* = \frac{1}{1+\delta^{1/\rho}(1-p)^{1/\rho}}\omega$ and $c_1^* = \frac{\delta^{1/\rho}(1-p)^{1/\rho}}{1+\delta^{1/\rho}(1-p)^{1/\rho}}\omega$.

2.2 In (1), set $r = 1$, $(c_t^*)^{-\rho} = \delta(1-p)(c_{t+1}^*)^{-\rho}$ or $c_{t+1}^* = c_t^*\delta^{1/\rho}(1-p)^{1/\rho} = c_0^*\delta^{(t+1)/\rho}(1-p)^{(t+1)/\rho}$. Therefore, $c_0^* = (1-\delta^{1/\rho}(1-p)^{1/\rho})\omega$.

2.3 For $\rho = 1$, $c_{t+1}^* = c_0^*\delta^{t+1}(1-p)^{t+1}$ and $c_0^* = (1-\delta(1-p))\omega$.

2.4 We guess that solution takes form $c = \alpha\omega$ and plug it into (1) to get

$$\frac{1}{\alpha c} = \delta(1-p)\mathbb{E}\left[\frac{r}{\alpha r\omega(1-\alpha)}\right],$$

and so, $c_t^* = (1-\delta(1-p))\omega_t$ where $\omega_t = r_t(\omega_{t-1} - c_{t-1})$ and $\omega_0 = \omega$.

3. I denote by $\succeq$ first-order stochastic dominance relationship, and by $\geq$ second-order stochastic dominance relationship.

3.1 True. $P(g(X) \leq t) = P(X \leq g^{-1}(t)) \leq P(Y \leq g^{-1}(t)) = P(g(Y) \leq t)$.

3.2 False. Consider $X = \frac{1}{2}$ and Y is uniform on $[0, 1]$. Then $X \geq Y$. Consider $g(t) = t^2$ and $u(t) = t$, then $\mathbb{E}u(g(Y)) = \mathbb{E}g(Y) > g(\mathbb{E}Y) = g(X) = \mathbb{E}u(g(X))$.

3.3 False. Consider the following counter-example. Let $\alpha = \frac{1}{2}$, X is uniform on $[0, 1]$ and $Y = 1 - X$. Since X and Y have the same distribution, $X \succeq Y$. However, $\frac{X+Y}{2} = \frac{1}{2}$ and X are not ordered according to $\succeq$.

4. Denote the share invested in asset $i = 1, 2$ by z_i. Optimal portfolio solves the problem

$$\max_{z_1,z_2} \mathbb{E} - e^{-\alpha(\omega+z_1(X-1)+z_2(Y-1))} = \max_{z_1,z_2} \omega + z_1(\mu-1) - \frac{z_1^2}{2}\alpha\sigma^2) + z_2(2\mu-1) - \frac{z_2^2}{2}\alpha\sigma^2)$$

which has solution $z_1 = \frac{1}{\alpha}\frac{\mu-1}{\sigma^2}$ and $z_2 = \frac{1}{\alpha}\frac{2\mu-1}{\sigma^2}$.