14.123 Problem Set 1 Solution

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Problem 1

(a) Apply iterated strict dominance. Note that d is strictly dominated by a mixture of $\frac{2}{5}a$ and $\frac{3}{5}b$, and z is strictly dominated by a mixture of $\frac{3}{5}w$ and $\frac{2}{5}x$. The other strategies are not strictly dominated. Therefore, $S^1 = \{a, b, c\} \times \{w, x, y\}$. In addition, $S^1 = S^2 = S^\infty$, because (for example) a, b, and c are all best responses to y; and w, x, and y are all best responses to c.

(b) Applying the definition of correlated equilibrium, it follows that a probability distribution $p \in \Delta(S)$ is a correlated equilibrium distribution if and only if

$$\begin{aligned}
3p((a,w)) &\geq 2p((a,x)) \\
2p((b,x)) &\geq 3p((b,w)) \\
2p((a,w)) &\geq 3p((b,w)) \\
3p((b,x)) &\geq 2p((a,x)) \\
p((c,w)) &= p((c,x)) = p((a,y)) = p((b,y)) = 0 \\
p(s_1, s_2) &= 0 \text{ whenever } s_1 = d \text{ or } s_2 = z.
\end{aligned}$$

The first four conditions above can be written more compactly as

$$\min\{p((a,w)), p((b,x))\} \geq \max\left\{\frac{2}{3}p((a,x)), \frac{3}{2}p((b,w))\right\}.$$

(c) The probability distribution $p \in \Delta(S)$ given by $p((a,w)) = p((b,x)) = \frac{1}{2}$ is a correlated equilibrium distribution by part (b) but clearly is not a Nash equilibrium distribution.

Problem 2

Aside: There was some confusion about this problem. The point of the problem is to show that every correlated equilibrium is a Bayesian Nash equilibrium of a closely related Bayesian game (in which players' types are like their information partitions in the correlated equilibrium information structure), and that every Bayesian Nash equilibrium of a Bayesian game where types do not affect players' payoffs is a correlated equilibrium with a closely related information structure (in which players' information partitions are like their types in the Bayesian game). Showing this requires working through the mathematical definitions of correlated equilibrium and Bayesian Nash equilibrium, and in particular requires keeping straight what objects characterize an information structure (i.e., the triple (Ω, I, p)) and a type space in a Bayesian game (i.e., the pair (T, p')). To be concrete, for part (a) you have to define a type space (T, p'), and for part (b) you have to define an information structure (Ω, I, p).

(a) Let $T = I$, so for each player $i \in N$ a type t_i of player i is an element of player i's information partition I_i. Let p' be given by

$$p'(t) = \sum_{\omega \in \Omega : I_i(\omega) = t_i \text{ for all } i \in N} p(\omega).$$

Let $\tau_i : I \to T$ be the identity map Id.

I must verify that an adapted strategy profile $\mathbf{s} : \Omega \to S$ with respect to (Ω, I, p) is a correlated equilibrium if and only if $\mathbf{s} \circ \tau^{-1} : T \to S$ is a Bayesian Nash equilibrium of (G, T, p'). To see this, note that for any player $i \in N$, state $\omega \in \Omega$, and strategy $s_i \in S_i$,

$$\begin{aligned} E\left[u_i\left(s_i, \mathbf{s}_{-i}(\omega')\right) | I_i(\omega)\right] &= \sum_{\omega' \in \Omega : I_i(\omega') = I_i(\omega)} u_i\left(s_i, \mathbf{s}_{-i}(\omega')\right) p(\omega') \\ &= \sum_{t' \in T : t'_i = t_i} u_i\left(s_i, \mathbf{s}_{-i}(t')\right) p'(t') \\ &= E\left[u_i\left(s_i, \mathbf{s}_{-i}(t')\right) | t_i\right]. \end{aligned}$$

Recall that $\mathbf{s}$ is a correlated equilibrium with respect to (Ω, I, p) if and only if $s_i(\omega) \in \arg\max_{s_i \in S_i} E\left[u_i\left(s_i, \mathbf{s}_{-i}(\omega')\right) | I_i(\omega)\right]$ for all $i \in N$ and $\omega \in \Omega$, and $\mathbf{s}$ is a Bayesian Nash equilibrium of (G, T, p') if and only if and only if $s_i(t_i) \in \arg\max_{s_i \in S_i} E\left[u_i\left(s_i, \mathbf{s}_{-i}(t')\right) | t_i\right]$

for all $i \in N$ and $t_i \in T_i$. Since I have shown that these conditions are equivalent, $\mathbf{s}$ is a correlated equilibrium with respect to (Ω, I, p) if and only if it is a Bayesian Nash equilibrium of (G, T, p').

(b) Let $\Omega = T$. Let I_i be the partition of Ω with elements of the form $t_i \times T_{-i}$; that is, for every state $t \in \Omega$, we have $I_i(t) = t_i \times T_{-i}$. Let $p = p'$ (that is, $p(t) = p'(t)$ for all $t \in T$). Let $w : T \to \Omega$ be the identity map Id.

To verify that a strategy profile $\mathbf{s} : T \to S$ is a Bayesian Nash equilibrium of (G, T, p') if and only if $\mathbf{s} \circ w^{-1} : \Omega \to S$ is a correlated equilibrium with respect to (Ω, I, p), note that for any player $i \in N$, type $t_i \in T_i$, and strategy $s_i \in S_i$,

$$\begin{aligned} E\left[u_i\left(s_i, \mathbf{s}_{-i}\left(t'\right)\right)|t_i\right] &= \sum_{t' \in T : t'_i = t_i} u_i\left(s_i, \mathbf{s}_{-i}\left(t'\right)\right) p'\left(t'\right) \\ &= \sum_{t' \in T : I_i(t') = I_i(t)} u_i\left(s_i, \mathbf{s}_{-i}\left(t'\right)\right) p'\left(t'\right) \\ &= E\left[u_i\left(s_i, \mathbf{s}_{-i}\left(t'\right)\right)|I_i\left(t\right)\right]. \end{aligned}$$

The result now follows from the definition of correlated equilibrium and Bayesian Nash equilibrium, as in part (a).

Problem 3

Aside: A major problem in part (a) was people showing that $Z \subseteq S^1$ rather than $Z \subseteq S^\infty$. It is crucial that you understand the difference between S^1 and S^∞. In particular, the set of strategies that are not strictly dominated is S^1, while S^∞ is in general strictly smaller. Therefore, the fact that every strategy in a CURB ("closed under rational behavior") set is not strictly dominated does *not* imply that every CURB set is contained in S^∞.

Also, if I were being pickier I would have taken points off of everyone's problem set for stating without proof in part (c) that every strategy $s_i \in S_i^\infty$ is not strictly dominated by a mixed strategy with support in S_i^∞. Recall that S_i^∞ is defined as $\bigcap_{m=0}^\infty S_i^m$, which does not imply this property. In fact, this property fails unless S is finite (or satisfies a similar property, like compactness); in this case, the statement in part (c) is false, so your proof of part (c) should have invoked the (implicit) assumption that S is finite. For your enjoyment (*not* for the problem set), here's a counterexample for infinite (and non-compact) S:

Let $N = 2$, $S_i = \mathbb{N} \cup \{x, y\}$ for $i = \{1, 2\}$ (where $\mathbb{N}$ is the natural numbers), and define the symmetric utility function u_i as follows: If $n_1, n_2 \in \mathbb{N}$, then

$$u_1(n_1, n_2) = \left\{ \begin{array}{c} 1 \text{ if } n_1 > n_2 \\ \frac{n_1 - 1}{n_1} \text{ if } n_1 \leq n_2 \end{array} \right\}.$$

If $n_2 \in \mathbb{N}$, then

$$u_1(x, n_2) = u_1(y, n_2) = 1.$$

Finally

$$\begin{aligned} u_1(x, x) &= u_1(x, y) = 1 \\ u_1(y, x) &= u_1(y, y) = 0. \end{aligned}$$

It can be easily checked that S_i^m is the set of integers greater than m, along with x and y. Therefore, $S_i^\infty = \{x, y\}$. But $\{x, y\}$ is *not* a CURB set, because x does strictly better than y against both x and y.

Now, on to the problem:

(a) I show that $Z \subseteq S^m$ for all $m \in \mathbb{N}$, by induction on m. Trivially, $Z \subseteq S^0$. Suppose $Z \subseteq S^m$ for some $m \in \mathbb{N}$, and fix $z_i \in Z_i$. Then there exists $\mu \in \Delta(Z_{-i}) \subseteq \Delta\left(S_{-1}^m\right)$ such that $z_i \in \arg\max_{s_i \in S_i} u_i(s_i, \mu)$. This implies that $z_i \in S^{m+1}$, by definition of S^{m+1}. Hence, $Z \subseteq S^{m+1}$, and therefore $Z \subseteq S^m$ for all $m \in \mathbb{N}$.

Since $Z \subseteq S^m$ for all $m \in \mathbb{N}$, it follows that $Z \subseteq \bigcap_{m=0}^\infty S^m = S^\infty$.

(b) Fix $i \in N$ and $z_i \in Z_i$. By definition of Z_i, there exists a CURB set Z_i^α such that $z_i \in Z_i^\alpha$. Because Z^α is CURB, there exists $\mu \in \Delta\left(Z_{-i}^\alpha\right) \subseteq \Delta(Z_{-i})$ such that $z_i \in \arg\max_{s_i \in S_i} u_i(s_i, \mu)$. Therefore, Z is CURB.

Let $Z^* \equiv \bigcup \{Z : Z \text{ is CURB}\}$ (I ignore the set-theoretic niceties here). We have established that Z^* is CURB, and Z^* contains any other CURB set by definition. Therefore, Z^* is the largest CURB set.

(c) By (a) and the fact that Z^* is CURB, we need only show that $S^\infty \subseteq Z^*$.

Assume that G is finite. Since $S^{m+1} \subseteq S^m$ for all $m \in \mathbb{N}$, this implies that there exists m^* such that $S^{m^*} = S^\infty$. By definition of S^∞, this implies that no strategy in $S_i^{m^*}$ is strictly dominated by a mixed strategy with support contained in $S_i^{m^*}$. By the theorem from lecture establishing equivalence of rational and undominated strategies, it follows that for every $i \in N$ and $s_i \in S_i^{m^*}$, there exists a belief $\mu \in \Delta\left(S_{-i}^{m^*}\right)$ such that $s_i \in \arg\max_{s_i' \in S_i} u_i\left(s_i', \mu\right)$. This implies that S^{m^*} is CURB, and the fact that $S^{m^*} = S^\infty$ now implies that S^∞ is CURB.

Having shown that S^∞ is CURB, it follows from the definition of Z^* that $S^\infty \subseteq Z^*$.