Handout on Externalities

1. Two roads, mandatory trips.

$F_i(N_i)$ is the cost for each person using road i if there are N_i people on road i, assumed to be nondecreasing in N, and increasing for some of the analysis.

There is a continuum of households of measure H.
Utility maximization:

$$\text{Max } u^h = I^h - \text{ Min } \left[F_1(N_1), F_2(N_2)\right] \tag{1.1}$$

Equilibrium assuming both roads are used in equilibrium:

$$\begin{aligned} &F_1(N_1) = F_2(N_2) \\ &N_1 + N_2 = H \end{aligned} \tag{1.2}$$

If only one road is used, then

$$F_1(H) \leq F_2(0) \tag{1.3}$$

Consider conditions for Pareto optimality for the subset of Pareto optima where everyone has positive income at the optimum.

Note that in this case, income distribution does not matter for optimal road use and we can derive the optimal road use by maximizing the sum of utilities.

There are other Pareto optima where these conditions do not hold – where all of the income goes to a subset of the people who get all the income and may also get a “better” road.

$$\begin{aligned} &\text{Max} \qquad \sum_h I^h - N_1 F_1(N_1) - N_2 F_2(N_2) \\ &\text{subject to } \; N_1 + N_2 = H \end{aligned} \tag{1.4}$$

If both roads are used, we have the FOC:

$$F_1(N_1)+N_1F_1'(N_1)=F_2(N_2)+N_2F_2'(N_2) \tag{1.5}$$

If just one road is used:

$$F_1(H)+HF_1'(H)\leq F_2(0) \tag{1.6}$$

In general the equilibrium is not Pareto optimal.

However, it can be Pareto optimal, for example, when only one road is used and when there is symmetry –

$$F_1(H/2)=F_2(H/2) \text{ and } F_1(H/2)+F_1'(H/2)H/2=F_2(H/2)+F_2'(H/2)H/2 \tag{1.7}$$

When equilibrium is not Pareto optimal, optimality can be restored either by taxing one of the roads or by subsidizing the other, with the revenue (or subsidy cost) covered by lump-sum taxes. When there are taxes on each road the equilibrium becomes

$$\begin{aligned} F_1(N_1)+t_1&=F_2(N_2)+t_2 \\ N_1+N_2&=H \end{aligned} \tag{1.8}$$

For the optimal N_i equation 1.8 gives us a continuum of values of the two taxes that achieve the equilibrium. This does not extend to the next example.

2. Two roads, voluntary trips.

h distributed uniformly on $[0,1]$

utility of trip for household h is g^h, decreasing in h.

Utility maximization:

$$u^h = I^h + \text{Max}\left[0, g^h - F_1(N_1), g^h - F_2(N_2)\right] \tag{2.1}$$

Equilibrium assuming both roads are used in equilibrium:

$$\begin{aligned} &g^{N_1+N_2} = F_1(N_1) = F_2(N_2) \\ &N_1 + N_2 \le H \end{aligned} \tag{2.2}$$

Conditions for Pareto optimum for optima where everyone has positive income at the optimum.

$$\begin{aligned} &\text{Max} \qquad \int_0^1 I^h + \int_0^{N_1+N_2} g^h dh - N_1 F_1(N_1) - N_2 F_2(N_2) \\ &\text{subject to } \; N_1 + N_2 \le H \end{aligned} \tag{2.3}$$

If both roads are used and some people do not make the trip, we have the FOC:

$$g^{N_1+N_2} = F_1(N_1) + N_1 F_1'(N_1) = F_2(N_2) + N_2 F_2'(N_2) \tag{2.4}$$

The optimum can be achieved by particular taxes on each road:

$$\begin{aligned} &g^{N_1+N_2} = F_1(N_1) + t_1 = F_2(N_2) + t_2 \\ &N_1 + N_2 \le H \end{aligned} \tag{2.5}$$

3. Production Economy with consumption externalities.

Assume a linear technology.

To derive conditions for Pareto optimality:

$$\begin{aligned} &\text{Max } \sum_h \alpha^h u^h\left(x^h, x^{\sim h}\right) \\ &\text{s.t } p \cdot \sum_h x^h = A \end{aligned} \tag{3.1}$$

FOC:

$$\sum_h \alpha^h \frac{\partial u^h}{\partial x_j^i} = \lambda p_j \qquad \vee i, j \tag{3.2}$$

Rearranging terms:

$$\alpha^i \frac{\partial u^i}{\partial x_j^i} = \lambda p_j - \sum_{h \neq i} \alpha^h \frac{\partial u^h}{\partial x_j^i} \tag{3.3}$$

Assume good 1 has no externalities and is the numeraire:

$$\alpha^h \frac{\partial u^h}{\partial x_1^h} = \lambda p_1 = \lambda \quad \forall h \tag{3.4}$$

Dividing 3.3 by 3.4:

$$\frac{\partial u^i / \partial x_j^i}{\partial u^i / \partial x_1^i} = \frac{p_j}{p_1} - \sum_{h \neq i} \frac{\partial u^h / \partial x_j^i}{\partial u^h / \partial x_1^h} \tag{3.5}$$

Introducing taxes on transactions, with taxes potentially being different for different individuals:

$$t_j^i = -\sum_{h \neq i} \frac{\partial u^h / \partial x_j^i}{\partial u^h / \partial x_1^h} \tag{3.6}$$

With these taxes, we can decentralize. We can see this by comparing FOC for individual choice with conditions for PO.

$$\begin{aligned} &\text{Max } u^h\left(x^h, x^{-h}\right) \\ &\text{s.t} \quad \sum\nolimits_{\text{j}}\left(p_j + t_j^h\right)x_j^h = I^h \end{aligned} \tag{3.7}$$

FOC:

$$\partial u^h / \partial x_j^h = \lambda^h\left(p_j + t_j^h\right) = \lambda^h\left(p_j - \sum_{i \neq h} \frac{\partial u^i / \partial x_j^h}{\partial u^i / \partial x_1^i}\right) \tag{3.8}$$

$$\partial u^h / \partial x_1^h = \lambda^h p_1 \tag{3.9}$$

Using (3.3) and (3.9), income levels must then be chosen so that

$$\lambda = \alpha^h \lambda^h \qquad \forall h \tag{3.10}$$

4. Pure public good.

Now consider government controlled expenditures, G, which potentially enter every utility function. We continue to assume a linear technology. Conditions for Pareto optimality:

$$\begin{aligned} &\text{Max} \quad \sum \alpha^h u^h \left(x^h, G \right) \\ &\text{s.t} \quad\ \ p \cdot \sum x^h + p_G G = R \end{aligned} \tag{4.1}$$

$$\alpha^h \frac{\partial u^h}{\partial x_i^h} = \lambda p_i \tag{4.2}$$

$$\sum_h \alpha^h \frac{\partial u^h}{\partial G} = \lambda p_G \tag{4.3}$$

Dividing 4.3 by 4.2:

$$\sum_h \frac{\partial u^h / \partial G}{\partial u^h / \partial x_i^h} = \frac{p_G}{p_i} \tag{4.4}$$