14.06 Lecture Notes

Intermediate Macroeconomics

George-Marios Angeletos

MIT Department of Economics

Spring 2004

Chapter 2

The Solow Growth Model (and a look ahead)

2.1 Centralized Dictatorial Allocations

- In this section, we start the analysis of the Solow model by pretending that there is a benevolent dictator, or social planner, that chooses the static and intertemporal allocation of resources and dictates that allocations to the households of the economy We will later show that the allocations that prevail in a decentralized competitive market environment coincide with the allocations dictated by the social planner.

2.1.1 The Economy, the Households and the Social Planner

- Time is discrete, $t \in \{0, 1, 2, ...\}$. You can think of the period as a year, as a generation, or as any other arbitrary length of time.

- The economy is an isolated island. Many households live in this island. There are no markets and production is centralized. There is a benevolent dictator, or social planner, who governs all economic and social affairs.

- There is one good, which is produced with two factors of production, capital and labor, and which can be either consumed in the same period, or invested as capital for the next period.

- Households are each endowed with one unit of labor, which they supply inelasticly to the social planner. The social planner uses the entire labor force together with the accumulated aggregate capital stock to produce the one good of the economy.

- In each period, the social planner saves a constant fraction $s \in (0, 1)$ of contemporaneous output, to be added to the economy's capital stock, and distributes the remaining fraction uniformly across the households of the economy.

- In what follows, we let L_t denote the number of households (and the size of the labor force) in period t, K_t aggregate capital stock in the beginning of period t, Y_t aggregate output in period t, C_t aggregate consumption in period t, and I_t aggregate investment in period t. The corresponding lower-case variables represent per-capita measures: $k_t = K_t/L_t$, $y_t = Y_t/L_t$, $i_t = I_t/L_t$, and $c_t = C_t/L_t$.

2.1.2 Technology and Production

- The technology for producing the good is given by

$$Y_t = F(K_t, L_t) \tag{2.1}$$

where $F : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ is a (stationary) production function. We assume that F is continuous and (although not always necessary) twice differentiable.

- We say that the technology is "*neoclassical*" if F satisfies the following properties

    1. Constant returns to scale (CRS), or linear homogeneity:

    $$F(\mu K, \mu L) = \mu F(K, L), \quad \forall \mu > 0.$$

    2. Positive and diminishing marginal products:

    $$F_K(K, L) > 0, \quad F_L(K, L) > 0,$$
    $$F_{KK}(K, L) < 0, \quad F_{LL}(K, L) < 0.$$

    where $F_x \equiv \partial F / \partial x$ and $F_{xz} \equiv \partial^2 F / (\partial x \partial z)$ for $x, z \in \{K, L\}$.

    3. Inada conditions:

    $$\lim_{K \to 0} F_K = \lim_{L \to 0} F_L = \infty,$$
    $$\lim_{K \to \infty} F_K = \lim_{L \to \infty} F_L = 0.$$

- By implication, F satisfies

$$Y = F(K, L) = F_K(K, L)K + F_L(K, L)L$$

or equivalently

$$1 = \varepsilon_K + \varepsilon_L$$

where

$$\varepsilon_K \equiv \frac{\partial F}{\partial K} \frac{K}{F} \quad \text{and} \quad \varepsilon_L \equiv \frac{\partial F}{\partial L} \frac{L}{F}$$

Also, F_K and F_L are homogeneous of degree zero, meaning that the marginal products depend only on the ratio K/L.

And, $F_{KL} > 0$, meaning that capital and labor are complementary.

- Technology in intensive form: Let

$$y = \frac{Y}{L} \quad \text{and} \quad k = \frac{K}{L}.$$

Then, by CRS

$$y = f(k) \tag{2.2}$$

where

$$f(k) \equiv F(k, 1).$$

By definition of f and the properties of F,

$$\begin{aligned} f(0) &= 0, \\ f'(k) &> 0 > f''(k) \\ \lim_{k\to 0} f'(k) &= \infty, \quad \lim_{k\to\infty} f'(k) = 0 \end{aligned}$$

Also,

$$\begin{aligned} F_K(K, L) &= f'(k) \\ F_L(K, L) &= f(k) - f'(k)k \end{aligned}$$

- The intensive-form production function f and the marginal product of capital f' are illustrated in **Figure 1**.

- *Example: Cobb-Douglas technology*

$$F(K, L) = K^\alpha L^{1-\alpha}$$

In this case,

$$\varepsilon_K = \alpha, \quad \varepsilon_L = 1 - \alpha$$

and

$$f(k) = k^\alpha.$$

2.1.3 The Resource Constraint, and the Law of Motions for Capital and Labor

- Remember that there is a single good, which can be either consumed or invested. Of course, the sum of aggregate consumption and aggregate investment can not exceed aggregate output. That is, the social planner faces the following *resource constraint*:

$$C_t + I_t \leq Y_t \tag{2.3}$$

Equivalently, in per-capita terms:

$$c_t + i_t \leq y_t \tag{2.4}$$

- Suppose that population growth is $n \geq 0$ per period. The size of the labor force then evolves over time as follows:

$$L_t = (1+n)L_{t-1} = (1+n)^t L_0 \tag{2.5}$$

We normalize $L_0 = 1$.

- Suppose that existing capital depreciates over time at a fixed rate $\delta \in [0, 1]$. The capital stock in the beginning of next period is given by the non-depreciated part of current-period capital, plus contemporaneous investment. That is, *the law of motion for capital* is

$$K_{t+1} = (1-\delta)K_t + I_t. \tag{2.6}$$

Equivalently, in per-capita terms:

$$(1+n)k_{t+1} = (1-\delta)k_t + i_t$$

We can approximately write the above as

$$k_{t+1} \approx (1 - \delta - n)k_t + i_t \tag{2.7}$$

The sum $\delta + n$ can thus be interpreted as the "effective" depreciation rate of per-capita capital. (Remark: This approximation becomes arbitrarily good as the economy converges to its steady state. Also, it would be exact if time was continuous rather than discrete.)

2.1.4 The Dynamics of Capital and Consumption

- In most of the growth models that we will examine in this class, the key of the analysis will be to derive a dynamic system that characterizes the evolution of aggregate consumption and capital in the economy; that is, a system of difference equations in C_t and K_t (or c_t and k_t). This system is very simple in the case of the Solow model.

- Combining the law of motion for capital (2.6), the resource constraint (2.3), and the technology (2.1), we derive the difference equation for the capital stock:

$$K_{t+1} - K_t = F(K_t, L_t) - \delta K_t - C_t \tag{2.8}$$

  That is, the change in the capital stock is given by aggregate output, minus capital depreciation, minus aggregate consumption.

$$k_{t+1} = f(k_t) - (\delta + n)k_t - c_t.$$

- On the other hand, aggregate consumption is, by assumption, a fixed fraction $(1 - s)$ of output:

$$C_t = (1 - s)F(K_t, L_t) \tag{2.9}$$

- Similarly, in per-capita terms, (2.6), (2.4) and (2.2) give the dynamics of capitalwhereas consumption is given by

$$c_t = (1 - s)f(k_t).$$

- *From this point and on, we will analyze the dynamics of the economy in per capita terms only.* Translating the results to aggregate terms is a straightforward exercise.

2.1.5 Feasible and "Optimal" Allocations

Definition 1 *A feasible allocation is any sequence* $\{c_t, k_t\}_{t=0}^{\infty}$ *that satisfies the resource constraint*

$$k_{t+1} = f(k_t) + (1 - \delta - n)k_t - c_t. \tag{2.10}$$

Definition 2 *An "optimal" centralized allocation is any feasible allocation that satisfies*

$$c_t = (1 - s)f(k_t). \tag{2.11}$$

- In the Ramsey model, the optimal allocation will maximize social welfare. Here, the "optimal" allocation satisfies the presumed rule-of-thump for the social planner.

2.1.6 The Policy Rule

- Combining (2.10) and (2.11), we derive *the fundamental equation of the Solow model*:

$$k_{t+1} - k_t = sf(k_t) - (\delta + n)k_t \tag{2.12}$$

Note that the above defines k_{t+1} as a function of k_t :

Proposition 3 *Given any initial point $k_0 > 0$, the dynamics of the dictatorial economy are given by the path $\{k_t\}_{t=0}^{\infty}$ such that*

$$k_{t+1} = G(k_t), \tag{2.13}$$

for all $t \geq 0$, where

$$G(k) \equiv sf(k) + (1 - \delta - n)k.$$

Equivalently, the growth rate of capital is given by

$$\gamma_t \equiv \frac{k_{t+1} - k_t}{k_t} = \gamma(k_t), \tag{2.14}$$

where

$$\gamma(k) \equiv s\phi(k) - (\delta + n), \quad \phi(k) \equiv f(k)/k.$$

- **Proof.** (2.13) follows from (2.12) and rearranging gives (2.14). **QED**

- G corresponds to what we will call the *policy rule* in the Ramsey model. The dynamic evolution of the economy is concisely represented by the path $\{k_t\}_{t=0}^{\infty}$ that satisfies (2.12), or equivalently (2.13), for all $t \geq 0$, with k_0 historically given.

- The graph of G is illustrated in **Figure 2**.

2.1.7 Steady State

- A *steady state* of the economy is defined as any level k^* such that, if the economy starts with $k_0 = k^*$, then $k_t = k^*$ for all $t \geq 1$. That is, a steady state is any fixed point k^* of (2.12) or (2.13). Equivalently, a steady state is any fixed point (c^*, k^*) of the system (2.10)-(2.11).

- A trivial steady state is $c = k = 0$: There is no capital, no output, and no consumption. This would not be a steady state if $f(0) > 0$. We are interested for steady states at which capital, output and consumption are all positive and finite. We can easily show:

Proposition 4 *Suppose $\delta+n \in (0,1)$ and $s \in (0,1)$. A steady state $(c^*, k^*) \in (0,\infty)^2$ for the dictatorial economy exists and is unique. k^* and y^* increase with s and decrease with δ and n, whereas c^* is non-monotonic with s and decreases with δ and n. Finally, $y^*/k^* = (\delta+n)/s$.*

- **Proof.** k^* is a steady state if and only if it solves

$$0 = sf(k^*) - (\delta + n)k^*,$$

Equivalently

$$\frac{y^*}{k^*} = \phi(k^*) = \frac{\delta + n}{s} \tag{2.15}$$

where

$$\phi(k) \equiv \frac{f(k)}{k}.$$

The function ϕ gives the output-to-capital ratio in the economy. The properties of f imply that ϕ is continuous (and twice differentiable), decreasing, and satisfies the Inada conditions at $k = 0$ and $k = \infty$:

$$\phi'(k) = \frac{f'(k)k - f(k)}{k^2} = -\frac{F_L}{k^2} < 0,$$
$$\phi(0) = f'(0) = \infty \quad \text{and} \quad \phi(\infty) = f'(\infty) = 0,$$

where the latter follow from L'Hospital's rule. This implies that equation (2.15) has a solution if and only if $\delta + n > 0$ and $s > 0$. and the solution unique whenever it exists. The steady state of the economy is thus unique and is given by

$$k^* = \phi^{-1}\left(\frac{\delta + n}{s}\right).$$

Since $\phi' < 0$, k^* is a decreasing function of $(\delta + n)/s$. On the other hand, consumption is given by

$$c^* = (1 - s)f(k^*).$$

It follows that c^* decreases with $\delta + n$, but s has an ambiguous effect. **QED**

2.1.8 Transitional Dynamics

- The above characterized the (unique) steady state of the economy. Naturally, we are interested to know whether the economy will converge to the steady state if it starts away from it. Another way to ask the same question is whether the economy will eventually return to the steady state after an exogenous shock perturbs the economy and moves away from the steady state.

- The following uses the properties of G to establish that, in the Solow model, convergence to the steady is always ensured and is monotonic:

Proposition 5 *Given any initial $k_0 \in (0, \infty)$, the dictatorial economy converges asymptotically to the steady state. The transition is monotonic. The growth rate is positive and decreases over time towards zero if $k_0 < k^*$; it is negative and increases over time towards zero if $k_0 > k^*$.*

- **Proof.** From the properties of f, $G'(k) = sf'(k) + (1 - \delta - n) > 0$ and $G''(k) = sf''(k) < 0$. That is, G is strictly increasing and strictly concave. Moreover, $G(0) = 0$, $G'(0) = \infty$, $G(\infty) = \infty$, $G'(\infty) = (1 - \delta - n) < 1$. By definition of k^*, $G(k) = k$ iff $k = k^*$. It follows that $G(k) > k$ for all $k < k^*$ and $G(k) < k$ for all $k > k^*$. It follows that $k_t < k_{t+1} < k^*$ whenever $k_t \in (0, k^*)$ and therefore the sequence $\{k_t\}_{t=0}^{\infty}$ is strictly increasing if $k_0 < k^*$. By monotonicity, k_t converges asymptotically to some $\hat{k} \leq k^*$.

By continuity of G, $\hat{k}$ must satisfy $\hat{k} = G(\hat{k})$, that is $\hat{k}$ must be a fixed point of G. But we already proved that G has a unique fixed point, which proves that $\hat{k} = k^*$. A symmetric argument proves that, when $k_0 > k^*$, $\{k_t\}_{t=0}^{\infty}$ is stricttly decreasing and again converges asymptotically to k^*. Next, consider the growth rate of the capital stock. This is given by

$$\gamma_t \equiv \frac{k_{t+1} - k_t}{k_t} = s\phi(k_t) - (\delta + n) \equiv \gamma(k_t).$$

Note that $\gamma(k) = 0$ iff $k = k^*$, $\gamma(k) > 0$ iff $k < k^*$, and $\gamma(k) < 0$ iff $k > k^*$. Moreover, by diminishing returns, $\gamma'(k) = s\phi'(k) < 0$. It follows that $\gamma(k_t) < \gamma(k_{t+1}) < \gamma(k^*) = 0$ whenever $k_t \in (0, k^*)$ and $\gamma(k_t) > \gamma(k_{t+1}) > \gamma(k^*) = 0$ whenever $k_t \in (k^*, \infty)$. This proves that γ_t is positive and decreases towards zero if $k_0 < k^*$ and it is negative and increases towards zero if $k_0 > k^*$. **QED**

- **Figure 2** depicts $G(k)$, the relation between k_t and k_{t+1}. The intersection of the graph of G with the 45$^{\text{o}}$ line gives the steady-state capital stock k^*. The arrows represent the path $\{k_t\}_{t=}^{\infty}$ for a particular initial k_0.

- **Figure 3** depicts $\gamma(k)$, the relation between k_t and γ_t. The intersection of the graph of γ with the 45$^{\text{o}}$ line gives the steady-state capital stock k^*. The negative slope reflects what we call "conditional convergence."

- Discuss local versus global stability: Because $\phi'(k^*) < 0$, the system is locally stable. Because ϕ is globally decreasing, the system is globally stable and transition is monotonic.

2.2 Decentralized Market Allocations

- In the previous section, we characterized the centralized allocations dictated by a social planner. We now characterize the allocations

2.2.1 Households

- Households are dynasties, living an infinite amount of time. We index households by $j \in [0,1]$, having normalized $L_0 = 1$. The number of heads in every household grow at constant rate $n \geq 0$. Therefore, the size of the population in period t is $L_t = (1+n)^t$ and the number of persons in each household in period t is also L_t.

- We write $c_t^j, k_t^j, b_t^j, i_t^j$ for the per-head variables for household j.

- Each person in a household is endowed with one unit of labor in every period, which he supplies inelasticly in a competitive labor market for the contemporaneous wage w_t. Household j is also endowed with initial capital k_0^j. Capital in household j accumulates according to

$$(1+n)k_{t+1}^j = (1-\delta)k_t^j + i_t,$$

  which we approximate by

$$k_{t+1}^j = (1-\delta-n)k_t^j + i_t. \tag{2.16}$$

  Households rent the capital they own to firms in a competitive rental market for a (gross) rental rate r_t.

- The household may also hold stocks of some firms in the economy. Let π_t^j be the dividends (firm profits) that household j receive in period t. As it will become clear later on, it is without any loss of generality to assume that there is no trade of stocks.

(This is because the value of firms stocks will be zero in equilibrium and thus the value of any stock transactions will be also zero.) We thus assume that household j holds a fixed fraction α^j of the aggregate index of stocks in the economy, so that $\pi_t^j = \alpha^j \Pi_t$, where Π_t are aggregate profits. Of course, $\int \alpha^j dj = 1$.

- The household uses its income to finance either consumption or investment in new capital:

$$c_t^j + i_t^j = y_t^j.$$

Total per-head income for household j in period t is simply

$$y_t^j = w_t + r_t k_t^j + \pi_t^j. \tag{2.17}$$

Combining, we can write the budget constraint of household j in period t as

$$c_t^j + i_t^j = w_t + r_t k_t^j + \pi_t^j \tag{2.18}$$

- Finally, the consumption and investment behavior of household is a simplistic linear rule. They save fraction s and consume the rest:

$$c_t^j = (1-s) y_t^j \quad \text{and} \quad i_t^j = s y_t^i. \tag{2.19}$$

2.2.2 Firms

- There is an arbitrary number M_t of firms in period t, indexed by $m \in [0, M_t]$. Firms employ labor and rent capital in competitive labor and capital markets, have access to the same neoclassical technology, and produce a homogeneous good that they sell competitively to the households in the economy.

- Let K_t^m and L_t^m denote the amount of capital and labor that firm m employs in period t. Then, the profits of that firm in period t are given by

$$\Pi_t^m = F(K_t^m, L_t^m) - r_t K_t^m - w_t L_t^m.$$

- The firms seek to maximize profits. The FOCs for an interior solution require

$$F_K(K_t^m, L_t^m) = r_t. \tag{2.20}$$

$$F_L(K_t^m, L_t^m) = w_t. \tag{2.21}$$

- Remember that the marginal products are homogenous of degree zero; that is, they depend only on the capital-labor ratio. In particular, F_K is a decreasing function of K_t^m/L_t^m and F_L is an increasing function of K_t^m/L_t^m. Each of the above conditions thus pins down a unique capital-labor ratio K_t^m/L_t^m. For an interior solution to the firms' problem to exist, it must be that r_t and w_t are consistent, that is, they imply the same K_t^m/L_t^m. This is the case if and only if there is some $X_t \in (0, \infty)$ such that

$$r_t = f'(X_t) \tag{2.22}$$

$$w_t = f(X_t) - f'(X_t)X_t \tag{2.23}$$

where $f(k) \equiv F(k, 1)$; this follows from the properties $F_K(K, L) = f'(K/L)$ and $F_L(K, L) = f(K/L) - f'(K/L) \cdot (K/L)$, which we established earlier.

- If (2.22) and (2.23) are satisfied, the FOCs reduce to $K_t^m/L_t^m = X_t$, or

$$K_t^m = X_t L_t^m. \tag{2.24}$$

That is, the FOCs pin down the capital labor ratio for each firm (K_t^m/L_t^m), but not the size of the firm (L_t^m). Moreover, because all firms have access to the same technology, they use exactly the same capital-labor ratio.

- Besides, (2.22) and (2.23) imply

$$r_t X_t + w_t = f(X_t). \tag{2.25}$$

It follows that

$$r_t K_t^m + w_t L_t^m = (r_t X_t + w_t) L_t^m = f(X_t) L_t^m = F(K_t^m, L_t^m),$$

and therefore

$$\Pi_t^m = L_t^m [f(X_t) - r_t X_t - w_t] = 0. \tag{2.26}$$

That is, when (2.22) and (2.23) are satisfied, the maximal profits that any firm makes are exactly zero, and these profits are attained for any firm size as long as the capital-labor ratio is optimal. If instead (2.22) and (2.23) were violated, then either $r_t X_t + w_t < f(X_t)$, in which case the firm could make infinite profits, or $r_t X_t + w_t > f(X_t)$, in which case operating a firm of any positive size would generate strictly negative profits.

2.2.3 Market Clearing

- The *capital market* clears if and only if

$$\int_0^{M_t} K_t^m dm = \int_0^1 (1+n)^t k_t^j dj$$

Equivalently,

$$\int_0^{M_t} K_t^m dm = K_t \tag{2.27}$$

where $K_t \equiv \int_0^{L_t} k_t^j dj$ is the aggregate capital stock in the economy.

- The *labor market*, on the other hand, clears if and only if

$$\int_0^{M_t} L_t^m dm = \int_0^1 (1+n)^t dj$$

Equivalently,

$$\int_0^{M_t} L_t^m dm = L_t \tag{2.28}$$

where L_t is the size of the labor force in the economy.

2.2.4 General Equilibrium: Definition

- The definition of a *general equilibrium* is more meaningful when households optimize their behavior (maximize utility) rather than being automata (mechanically save a constant fraction of income). Nonetheless, it is always important to have clear in mind what is the definition of equilibrium in any model. For the decentralized version of the Solow model, we let:

Definition 6 *An equilibrium of the economy is an allocation* $\{(k_t^j, c_t^j, i_t^j)_{j\in[0,1]}, (K_t^m, L_t^m)_{m\in[0,M_t]}\}_{t=0}^\infty$, *a distribution of profits* $\{(\pi_t^j)_{j\in[0,1]}\}$, *and a price path* $\{r_t, w_t\}_{t=0}^\infty$ *such that*

(i) Given $\{r_t, w_t\}_{t=0}^\infty$ *and* $\{\pi_t^j\}_{t=0}^\infty$, *the path* $\{k_t^j, c_t^j, i_t^j\}$ *is consistent with the behavior of household* j, *for every* j.

(ii) (K_t^m, L_t^m) *maximizes firm profits, for every* m *and* t.

(iii) The capital and labor markets clear in every period

(iv) Aggregate dividends equal aggregate profits.

2.2.5 General Equilibrium: Existence, Uniqueness, and Characterization

- In the next, we characterize the decentralized equilibrium allocations:

Proposition 7 *For any initial positions* $(k_0^j)_{j\in[0,1]}$, *an equilibrium exists. The allocation of production across firms is indeterminate, but the equilibrium is unique as regards aggregate* *and household allocations. The capital-labor ratio in the economy is given by* $\{k_t\}_{t=0}^{\infty}$ *such that*

$$k_{t+1} = G(k_t) \tag{2.29}$$

for all $t \geq 0$ *and* $k_0 = \int k_0^j dj$ *historically given, where* $G(k) \equiv sf(k)+(1-\delta-n)k$*. Equilibrium growth is given by*

$$\gamma_t \equiv \frac{k_{t+1} - k_t}{k_t} = \gamma(k_t), \tag{2.30}$$

where $\gamma(k) \equiv s\phi(k) - (\delta + n),\ \phi(k) \equiv f(k)/k$*. Finally, equilibrium prices are given by*

$$r_t = r(k_t) \equiv f'(k_t), \tag{2.31}$$

$$w_t = w(k_t) \equiv f(k_t) - f'(k_t)k_t, \tag{2.32}$$

where $r'(k) < 0 < w'(k)$*.*

- **Proof.** We first characterize the equilibrium, assuming it exists.

  Using $K_t^m = X_t L_t^m$ by (2.24), we can write the aggregate demand for capital as

  $$\int_0^{M_t} K_t^m dm = X_t \int_0^{M_t} L_t^m dm$$

  From the labor market clearing condition (2.28),

  $$\int_0^{M_t} L_t^m dm = L_t.$$

  Combining, we infer

  $$\int_0^{M_t} K_t^m dm = X_t L_t,$$

  and substituting in the capital market clearing condition (2.27), we conclude

  $$X_t L_t = K_t,$$

where $K_t \equiv \int_0^{L_t} k_t^j dj$ denotes the aggregate capital stock. Equivalently, letting $k_t \equiv K_t/L_t$ denote the capital-labor ratio in the economy, we have

$$X_t = k_t. \tag{2.33}$$

That is, all firms use the same capital-labor ratio as the aggregate of the economy.

Substituting (2.33) into (2.22) and (2.23) we infer that equilibrium prices are given by

$$\begin{aligned} r_t &= r(k_t) \equiv f'(k_t) = F_K(k_t, 1) \\ w_t &= w(k_t) \equiv f(k_t) - f'(k_t)k_t = F_L(k_t, 1) \end{aligned}$$

Note that $r'(k) = f''(k) = F_{KK} < 0$ and $w'(k) = -f''(k)k = F_{LK} > 0$. That is, the interest rate is a decreasing function of the capital-labor ratio and the wage rate is an increasing function of the capital-labor ratio. The first properties reflects diminishing returns, the second reflects the complementarity of capital and labor.

Adding up the budget constraints of the households, we get

$$C_t + I_t = r_t K_t + w_t L_t + \int \pi_t^j dj,$$

where $C_t \equiv \int c_t^j dj$ and $I_t \equiv \int i_t^j dj$. Aggregate dividends must equal aggregate profits, $\int \pi_t^j dj = \int \Pi_t^m dj$. By (2.26), profits for each firm are zero. Therefore, $\int \pi_t^j dj = 0$, implying

$$C_t + I_t = Y_t = r_t K_t + w_t L_t$$

Equivalently, in per-capita terms,

$$c_t + i_t = y_t = r_t k_t + w_t.$$

From (2.25) and (2.33), or equivalently from (2.31) and (2.32),

$$r_t k_t + w_t = y_t = f(k_t)$$

We conclude that the household budgets imply

$$c_t + i_t = f(k_t),$$

which is simply the resource constraint of the economy.

Adding up the individual capital accumulation rules (2.16), we get the capital accumulation rule for the aggregate of the economy. In per-capita terms,

$$k_{t+1} = (1 - \delta - n)k_t + i_t$$

Adding up (2.19) across households, we similarly infer

$$i_t = sy_t = sf(k_t).$$

Combining, we conclude

$$k_{t+1} = sf(k_t) + (1 - \delta - n)k_t = G(k_t),$$

which is exactly the same as in the centralized allocation.

Finally, existence and uniqueness is now trivial. (2.29) maps any $k_t \in (0, \infty)$ to a unique $k_{t+1} \in (0, \infty)$. Similarly, (2.31) and (2.32) map any $k_t \in (0, \infty)$ to unique $r_t, w_t \in (0, \infty)$. Therefore, given any initial $k_0 = \int k_0^j dj$, there exist unique paths $\{k_t\}_{t=0}^{\infty}$ and $\{r_t, w_t\}_{t=0}^{\infty}$. Given $\{r_t, w_t\}_{t=0}^{\infty}$, the allocation $\{k_t^j, c_t^j, i_t^j\}$ for any household j is then uniquely determined by (2.16), (2.17), and (2.19). Finally, any allocation $(K_t^m, L_t^m)_{m \in [0, M_t]}$ of production across firms in period t is consistent with equilibrium as long as $K_t^m = k_t L_t^m$. **QED**

- An immediate implication is that the decentralized market economy and the centralized dictatorial economy are isomorphic. This follows directly from the fact that G is the same under both regimes, provided of course that (s, δ, n, f) are the same:

Proposition 8 *The aggregate and per-capita allocations in the competitive market economy coincide with those in the dictatorial economy.*

- Given this isomorphism, we can immediately translate the steady state and the transitional dynamics of the centralized plan to the steady state and the transitional dynamics of the decentralized market allocations:

Corollary 9 *Suppose* $\delta + n \in (0,1)$ *and* $s \in (0,1)$. *A steady state* $(c^*, k^*) \in (0,\infty)^2$ *for the competitive economy exists and is unique, and coincides with that of the social planner.* k^* *and* y^* *increase with* s *and decrease with* δ *and* n, *whereas* c^* *is non-monotonic with* s *and decreases with* δ *and* n. *Finally,* $y^*/k^* = (\delta + n)/s$.

Corollary 10 *Given any initial* $k_0 \in (0,\infty)$, *the competitive economy converges asymptotically to the steady state. The transition is monotonic. The equilibrium growth rate is positive and decreases over time towards zero if* $k_0 < k^*$; *it is negative and increases over time towards zero if* $k_0 > k^*$.

2.3 Shocks and Policies

- The Solow model can be interpreted also as a primitive Real Business Cycle (RBC) model. We can use the model to predict the response of the economy to productivity or taste shocks, or to shocks in government policies.

2.3.1 Productivity (or Taste) Shocks

- Suppose output is given by

$$Y_t = F(K_t, L_t)$$

or in intensive form

$$y_t = A_t f(k_t)$$

where A_t denotes total factor productivity.

- Consider a permanent negative shock in productivity. The $G(k)$ and $\gamma(k)$ functions shift down, as illustrated in **Figure 4.** The new steady state is lower. The economy transits slowly from the old steady state to the new.

- If instead the shock is transitory, the shift in $G(k)$ and $\gamma(k)$ is also temporary. Initially, capital and output fall towards the low steady state. But when productivity reverts to the initial level, capital and output start to grow back towards the old high steady state.

- The effect of a prodictivity shock on k_t and y_t is illustrated in **Figure 5**. The solid lines correspond to a transitory shock, whereas the dashed lines correspond to a permanent shock.

- *Taste shocks:* Consider a temporary fall in the saving rate s. The $\gamma(k)$ function shifts down for a while, and then return to its initial position. What are the transitional dynamics? What if instead the fall in s is permanent?

2.3.2 Unproductive Government Spending

- Let us now introduce a *government* in the competitive market economy. The government spends resources without contributing to production or capital accumulation.

- The resource constraint of the economy now becomes

$$c_t + g_t + i_t = y_t = f(k_t),$$

where g_t denotes government consumption. It follows that the dynamics of capital are given by

$$k_{t+1} - k_t = f(k_t) - (\delta + n)k_t - c_t - g_t$$

- Government spending is financed with proportional income taxation, at rate $\tau \geq 0$. The government thus absorbs a fraction τ of aggregate output:

$$g_t = \tau y_t.$$

- Disposable income for the representative household is $(1-\tau)y_t$. We continue to assume that consumption and investment absorb fractions $1-s$ and s of disposable income:

$$\begin{aligned} c_t &= (1-s)(y_t - g_t), \\ i_t &= (1-s)(y_t - g_t). \end{aligned}$$

- Combining the above, we conclude that the dynamics of capital are now given by

$$\gamma_t = \frac{k_{t+1} - k_t}{k_t} = s(1-\tau)\phi(k_t) - (\delta + n).$$

where $\phi(k) \equiv f(k)/k$. Given s and k_t, the growth rate γ_t decreases with τ.

- A steady state exists for any $\tau \in [0, 1)$ and is given by

$$k^* = \phi^{-1}\left(\frac{\delta + n}{s(1-\tau)}\right).$$

Given s, k^* decreases with τ.

- *Policy Shocks:* Consider a temporary shock in government consumption. What are the transitional dynamics?

2.3.3 Productive Government Spending

- Suppose now that production is given by

$$y_t = f(k_t, g_t) = k_t^\alpha g_t^\beta,$$

where $\alpha > 0$, $\beta > 0$, and $\alpha + \beta < 1$. Government spending can thus be interpreted as infrastructure or other productive services. The resource constraint is

$$c_t + g_t + i_t = y_t = f(k_t, g_t).$$

- We assume again that government spending is financed with proportional income taxation at rate τ, and that private consumption and investment are fractions $1-s$ and s of disposable household income:

$$\begin{aligned} g_t &= \tau y_t. \\ c_t &= (1-s)(y_t - g_t) \\ i_t &= s(y_t - g_t) \end{aligned}$$

- Substituting $g_t = \tau y_t$ into $y_t = k_t^\alpha g_t^\beta$ and solving for y_t, we infer

$$y_t = k_t^{\frac{\alpha}{1-\beta}} \tau^{\frac{\beta}{1-\beta}} \equiv k_t^a \tau^b$$

where $a \equiv \alpha/(1-\beta)$ and $b \equiv \beta/(1-\beta)$. Note that $a > \alpha$, reflecting the complementarity between government spending and capital.

- We conclude that the growth rate is given by

$$\gamma_t = \frac{k_{t+1} - k_t}{k_t} = s(1-\tau)\tau^b k_t^{a-1} - (\delta + n).$$

The steady state is

$$k^* = \left(\frac{s(1-\tau)\tau^b}{\delta + n}\right)^{1/(1-a)}.$$

- Consider the rate τ that maximizes either k^*, or γ_t for any given k_t. This is given by

$$\frac{d}{d\tau}[(1-\tau)\tau^b] = 0 \Leftrightarrow$$
$$b\tau^{b-1} - (1+b)\tau^b = 0 \Leftrightarrow$$
$$\tau = b/(1+b) = \beta.$$

That is, the growth-maximizing τ equals the elasticity of production with respect to government services. The more productive government services are, the higher their "optimal" provision.

2.4 Continuous Time and Conditional Convergence

2.4.1 The Solow Model in Continuous Time

- Recall that the basic growth equation in the discrete-time Solow model is

$$\frac{k_{t+1} - k_t}{k_t} = \gamma(k_t) \equiv s\phi(k_t) - (\delta + n).$$

We would expect a similar condition to hold under continuous time. We verify this below.

- The resource constraint of the economy is

$$C + I = Y = F(K, L).$$

In per-capita terms,

$$c + i = y = f(k).$$

- Population growth is now given by

$$\frac{\dot{L}}{L} = n$$

and the law of motion for aggregate capital is

$$\dot{K} = I - \delta K$$

- Let $k \equiv K/L$. Then,

$$\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{L}}{L}.$$

Substituting from the above, we infer

$$\dot{k} = i - (\delta + n)k.$$

Combining this with

$$i = sy = sf(k),$$

we conclude

$$\dot{k} = sf(k) - (\delta + n)k.$$

- Equivalently, the growth rate of the economy is given by

$$\frac{\dot{k}}{k} = \gamma(k) \equiv s\phi(k) - (\delta + n). \tag{2.34}$$

The function $\gamma(k)$ thus gives the growth rate of the economy in the Solow model, whether time is discrete or continuous.

2.4.2 Log-linearization and the Convergence Rate

- Define $z \equiv \ln k - \ln k^*$. We can rewrite the growth equation (2.34) as

$$\dot{z} = \Gamma(z),$$

where

$$\Gamma(z) \equiv \gamma(k^* e^z) \equiv s\phi(k^* e^z) - (\delta + n)$$

Note that $\Gamma(z)$ is defined for all $z \in \mathbb{R}$. By definition of k^*, $\Gamma(0) = s\phi(k^*) - (\delta + n) = 0$. Similarly, $\Gamma(z) > 0$ for all $z < 0$ and $\Gamma(z) < 0$ for all $z > 0$. Finally, $\Gamma'(z) = s\phi'(k^* e^z) k^* e^z < 0$ for all $z \in \mathbb{R}$.

- We next (log)linearize $\dot{z} = \Gamma(z)$ around $z = 0$:

$$\dot{z} = \Gamma(0) + \Gamma'(0) \cdot z$$

or equivalently

$$\dot{z} = \lambda z$$

where we substituted $\Gamma(0) = 0$ and let $\lambda \equiv \Gamma'(0)$.

- Straightforward algebra gives

$$\begin{aligned} \Gamma'(z) &= s\phi'(k^* e^z) k^* e^z < 0 \\ \phi'(k) &= \frac{f'(k)k - f(k)}{k^2} = -\left[1 - \frac{f'(k)k}{f(k)}\right] \frac{f(k)}{k^2} \\ sf(k^*) &= (\delta + n)k^* \end{aligned}$$

We infer

$$\Gamma'(0) = -(1 - \varepsilon_K)(\delta + n) < 0$$

where $\varepsilon_K \equiv F_K K / F = f'(k)k/f(k)$ is the elasticity of production with respect to capital, evaluated at the steady-state k.

- We conclude that

$$\frac{\dot{k}}{k} = \lambda \ln\left(\frac{k}{k^*}\right)$$

where

$$\lambda = -(1 - \varepsilon_K)(\delta + n) < 0$$

The quantity $-\lambda$ is called the *convergence rate*.

- In the Cobb-Douglas case, $y = k^\alpha$, the convergence rate is simply

$$-\lambda = (1 - \alpha)(\delta + n),$$

where α is the income share of capital. Note that as $\lambda \to 0$ as $\alpha \to 1$. That is, convergence becomes slower and slower as the income share of capital becomes closer and closer to 1. Indeed, if it were $\alpha = 1$, the economy would a balanced growth path.

- Note that, around the steady state

$$\frac{\dot{y}}{y} = \varepsilon_K \cdot \frac{\dot{k}}{k}$$

and

$$\frac{y}{y^*} = \varepsilon_K \cdot \frac{k}{k^*}$$

It follows that

$$\frac{\dot{y}}{y} = \lambda \ln \left(\frac{y}{y^*} \right)$$

Thus, $-\lambda$ is the convergence rate for either capital or output.

- In the example with productive government spending, $y = k^\alpha g^\beta = k^{\alpha/(1-\beta)} \tau^{\beta/(1-\beta)}$, we get

$$-\lambda = \left(1 - \frac{\alpha}{1 - \beta} \right) (\delta + n)$$

The convergence rate thus decreases with β, the productivity of government services. And $\lambda \to 0$ as $\beta \to 1 - \alpha$.

- *Calibration:* If $\alpha = 35\%$, $n = 3\%$ (= 1% population growth+2% exogenous technological process), and $\delta = 5\%$, then $-\lambda = 6\%$. This contradicts the data. But if $\alpha = 70\%$, then $-\lambda = 2.4\%$, which matches the data.

2.5 Cross-Country Differences and Conditional Convergence.

2.5.1 Mankiw-Romer-Weil: Cross-Country Differences

- The Solow model implies that steady-state capital, productivity, and income are determined primarily by technology (f and δ), the national saving rate (s), and population growth (n).

- Suppose that countries share the same technology in the long run, but differ in terms of saving behavior and fertility rates. If the Solow model is correct, observed cross-country income and productivity differences should be "explained" by observed cross-country differences in s and n,

- Mankiw, Romer and Weil tests this hypothesis against the data. In it's simple form, the Solow model fails to predict the large cross-country dispersion of income and productivity levels.

- Mankiw, Romer and Weil then consider an extension of the Solow model, that includes two types of capital, physical capital (k) and human capital (h). Output is given by

$$y = k^{\alpha} h^{\beta},$$

where $\alpha > 0, \beta > 0$, and $\alpha + \beta < 1$. The dynamics of capital accumulation are now given by

$$\begin{aligned} \dot{k} &= s_k y - (\delta + n)k \\ \dot{h} &= s_h y - (\delta + n)h \end{aligned}$$

where s_k and s_h are the investment rates in physical capital and human capital, respectively. The steady-state levels of k, h, and y then depend on both s_k and s_h, as well as δ and n.

- Proxying s_h by education attainment levels in each country, Mankiw, Romer and Weil find that the Solow model extended for human capital does a pretty good job in "explaining" the cross-country dispersion of output and productivity levels.

2.5.2 Barro: Conditional Convergence

- Recall the log-linearization of the dynamics around the steady state:

$$\frac{\dot{y}}{y} = \lambda \ln \frac{y}{y^*}.$$

A similar relation will hold true in the neoclassical growth model a la Ramsey-Cass-Koopmans. $\lambda < 0$ reflects local diminishing returns. Such local diminishing returns occur even in endogenous-growth models. The above thus extends well beyond the simple Solow model.

- Rewrite the above as

$$\Delta \ln y = \lambda \ln y - \lambda \ln y^*$$

Next, let us proxy the steady state output by a set of country-specific controls X, which include s, δ, n, τ etc. That is, let

$$-\lambda \ln y^* \approx \beta' X.$$

We conclude

$$\Delta \ln y = \lambda \ln y + \beta' X + error$$

- The above represents a typical "Barro-like" conditional-convergence regression: We use cross-country data to estimate λ (the convergence rate), together with β (the effects of the saving rate, education, population growth, policies, etc.) The estimated convergence rate is about 2% per year.

- Discuss the effects of the other variables (X).

2.6 Miscellaneous

2.6.1 The Golden Rule and Dynamic Inefficiency

- *The Golden Rule:* Consumption at the steady state is given by

$$\begin{aligned} c^* &= (1-s) f(k^*) = \\ &= f(k^*) - (\delta + n) k^* \end{aligned}$$

Suppose the social planner chooses s so as to maximize c^*. Since k^* is a monotonic function of s, this is equivalent to choosing k^* so as to maximize c^*. Note that

$$c^* = f(k^*) - (\delta + n) k^*$$

is strictly concave in k^*. The FOC is thus both necessary and sufficient. c^* is thus maximized if and only if $k^* = k_{gold}$, where k_{gold} solves

$$f'(k_{gold}) - \delta = n.$$

Equivalently, $s = s_{gold}$, where s_{gold} solves

$$s_{gold} \cdot \phi\left(k_{gold}\right) = (\delta + n)$$

The above is called the "*golden rule*" for savings, after Phelps.

- *Dynamic Inefficiency:* If $s > s_{gold}$ (equivalently, $k^* > k_{gold}$), the economy is dynamically inefficient: If the saving rate is lowered to $s = s_{gold}$ for all t, then consumption in all periods will be higher!

- On the other hand, if $s < s_{gold}$ (equivalently, $k^* > k_{gold}$), then raising s towards s_{gold} will increase consumption in the long run, but at the cost of lower consumption in the short run. Whether such a trade-off between short-run and long-run consumption is desirable will depend on how the social planner weight the short run versus the long run.

- *The Modified Golden Rule:* In the Ramsey model, this trade-off will be resolved when k^* satisfies the

$$f'(k^*) - \delta = n + \rho,$$

where $\rho > 0$ measures impatience (ρ will be called "the discount rate"). The above is called the "*modified golden rule.*" Naturally, the distance between the Ramsey-optimal k^* and the golden-rule k_{gold} increases with ρ.

- *Abel et. al.:* Note that the golden rule can be restated as

$$r - \delta = \frac{\dot{Y}}{Y}.$$

Dynamic inefficiency occurs when $r - \delta < \dot{Y}/Y$, dynamic efficiency is ensured if $r - \delta > \dot{Y}/Y$. Abel et al. use this relation to argue that, in the data, there is no evidence of dynamic inefficiency.

- *Bubbles:* If the economy is dynamically inefficient, there is room for bubbles.

2.6.2 Poverty Traps, Cycles, etc.

- The assumptions we have imposed on savings and technology implied that G is increasing and concave, so that there is a unique and globally stable steady state. More generally, however, G could be non-concave or even non-monotonic. Such "pathologies" can arise, for example, when the technology is non-convex, as in the case of locally increasing returns, or when saving rates are highly sensitive to the level of output, as in some OLG models.

- **Figure 6** illustrates an example of a non-concave G. There are now *multiple* steady states. The two extreme ones are (locally) stable, the intermediate is unstable versus stable ones. The lower of the stable steady states represents a *poverty trap*.

- **Figure 7** illustrates an example of a non-monotonic G. We can now have oscillating dynamics, or even perpetual endogenous cycles.

2.6.3 Introducing Endogenous Growth

- What ensures that the growth rate asymptotes to zero in the Solow model (and the Ramsey model as well) is the vanishing marginal product of capital, that is, the Inada condition $\lim_{k\to\infty} f'(k) = 0$.

- Continue to assume that $f''(k) < 0$, so that $\gamma'(k) < 0$, but assume now that $\lim_{k\to\infty} f'(k) = A > 0$. This implies also $\lim_{k\to\infty} \phi(k) = A$. Then, as $k \to \infty$,

$$\gamma_t \equiv \frac{k_{t+1} - k_t}{k_t} \to sA - (n + \delta)$$

- If $sA < (n+\delta)$, then it is like before: The economy converges to k^* such that $\gamma(k^*) = 0$. But if $sA > (n + x + \delta)$, then the economy exhibits deminishing but not vanishing growth: γ_t falls with t, but $\gamma_t \to sA - (n + \delta) > 0$ as $t \to \infty$.

- Jones and Manuelli consider such a general convex technology: e.g., $f(k) = Bk^\alpha + Ak$. We then get both transitional dynamics in the short run and perpetual growth in the long run.

- In case that $f(k) = Ak$, the economy follows a balanced-growth path from the very beginning.

- We will later "endogenize" A in terms of externalities, R&D, policies, institutions, markets, etc.

- For example, Romer/Lucas: If we have human capital or spillover effects,

$$y = Ak^\alpha h^{1-\alpha}$$

  and $h = k$, then we get $y = Ak$.

- Reconcile conditional convergence with endogenous growth. Think of $\ln k - \ln k^*$ as a detrended measure of the steady-state allocation of resources (human versus physical capital, specialization pattern.); or as a measure of distance from technology frontier; etc.

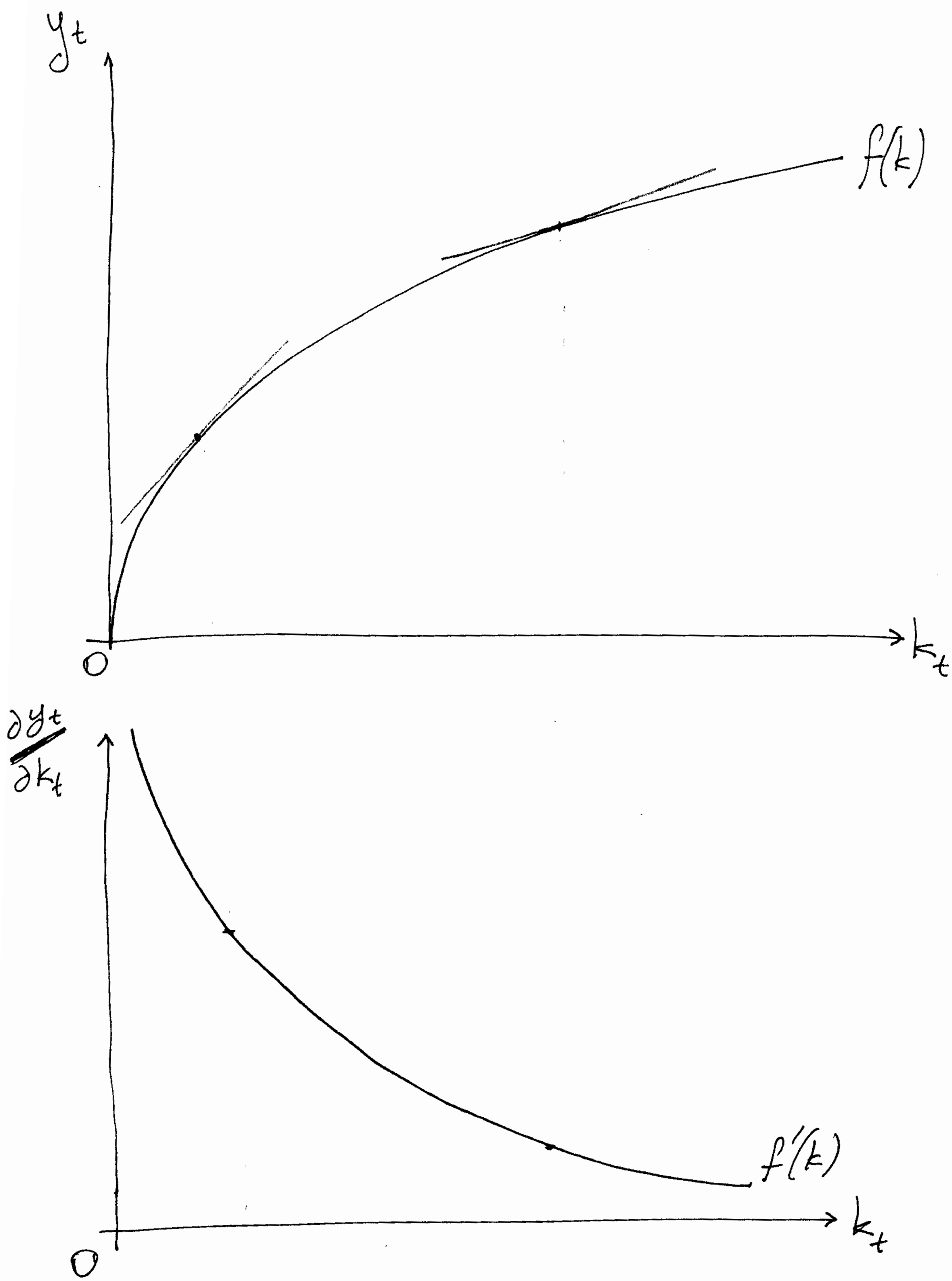

FIGURE 1

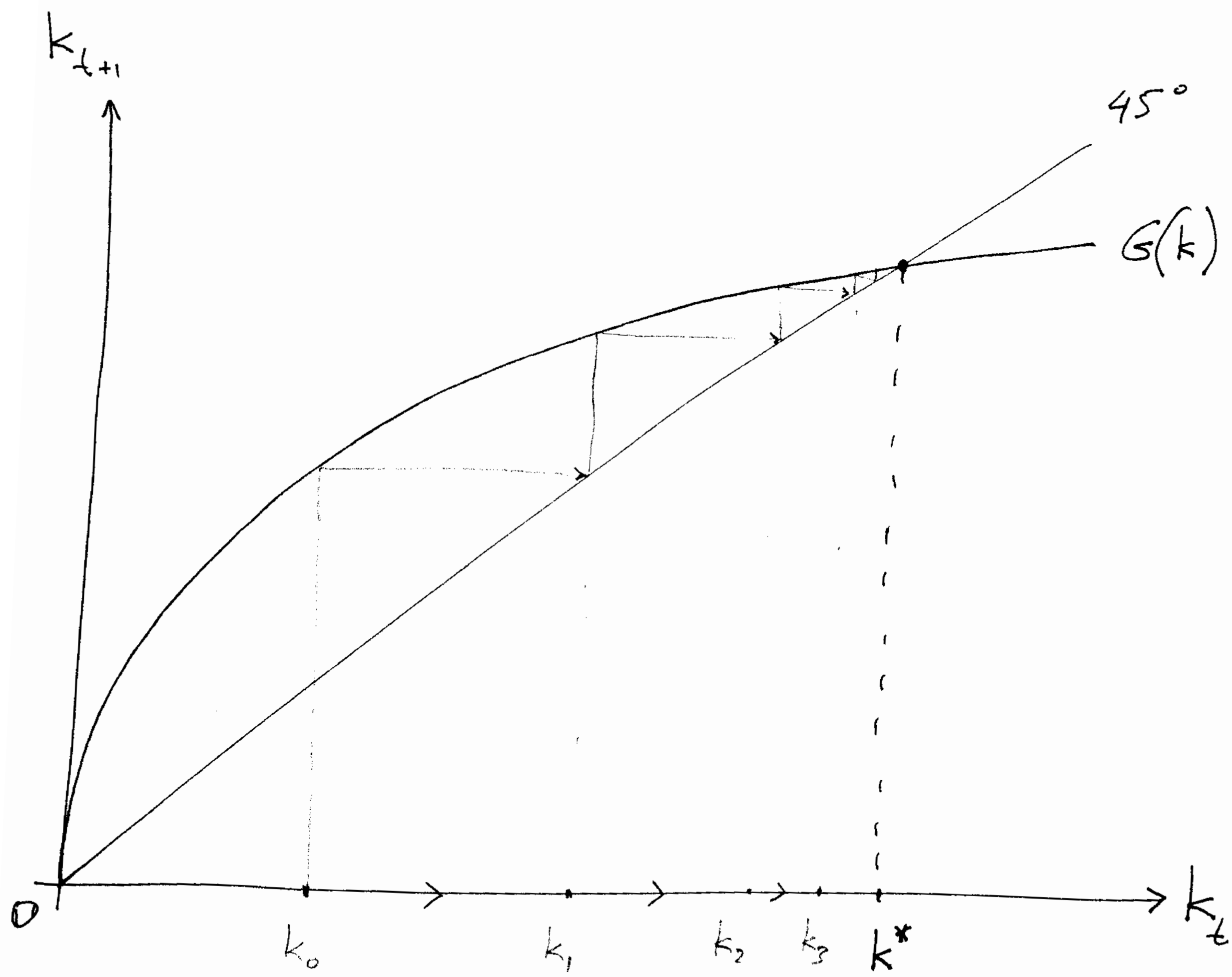

FIGURE 2.

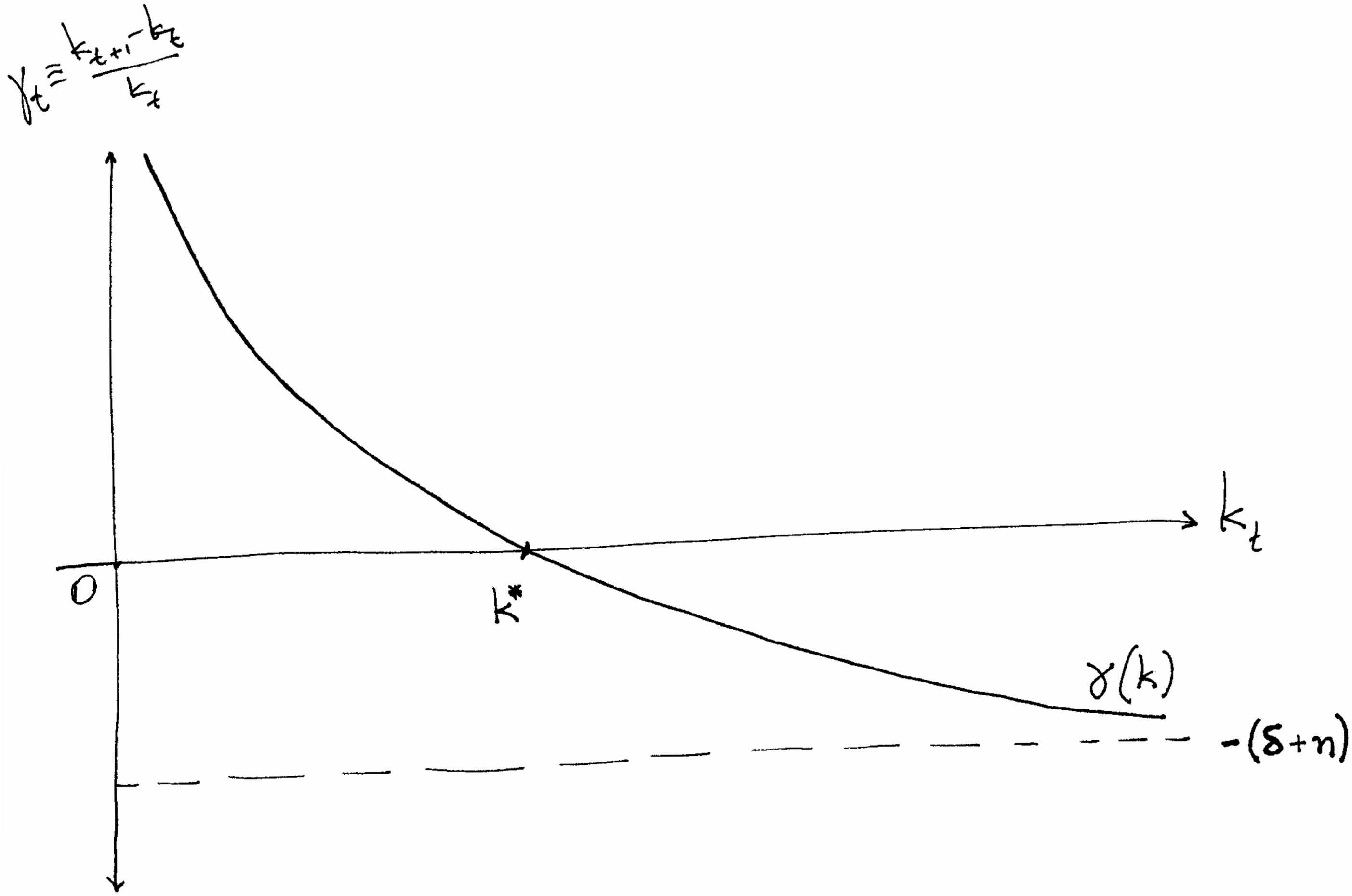

FIGURE 3

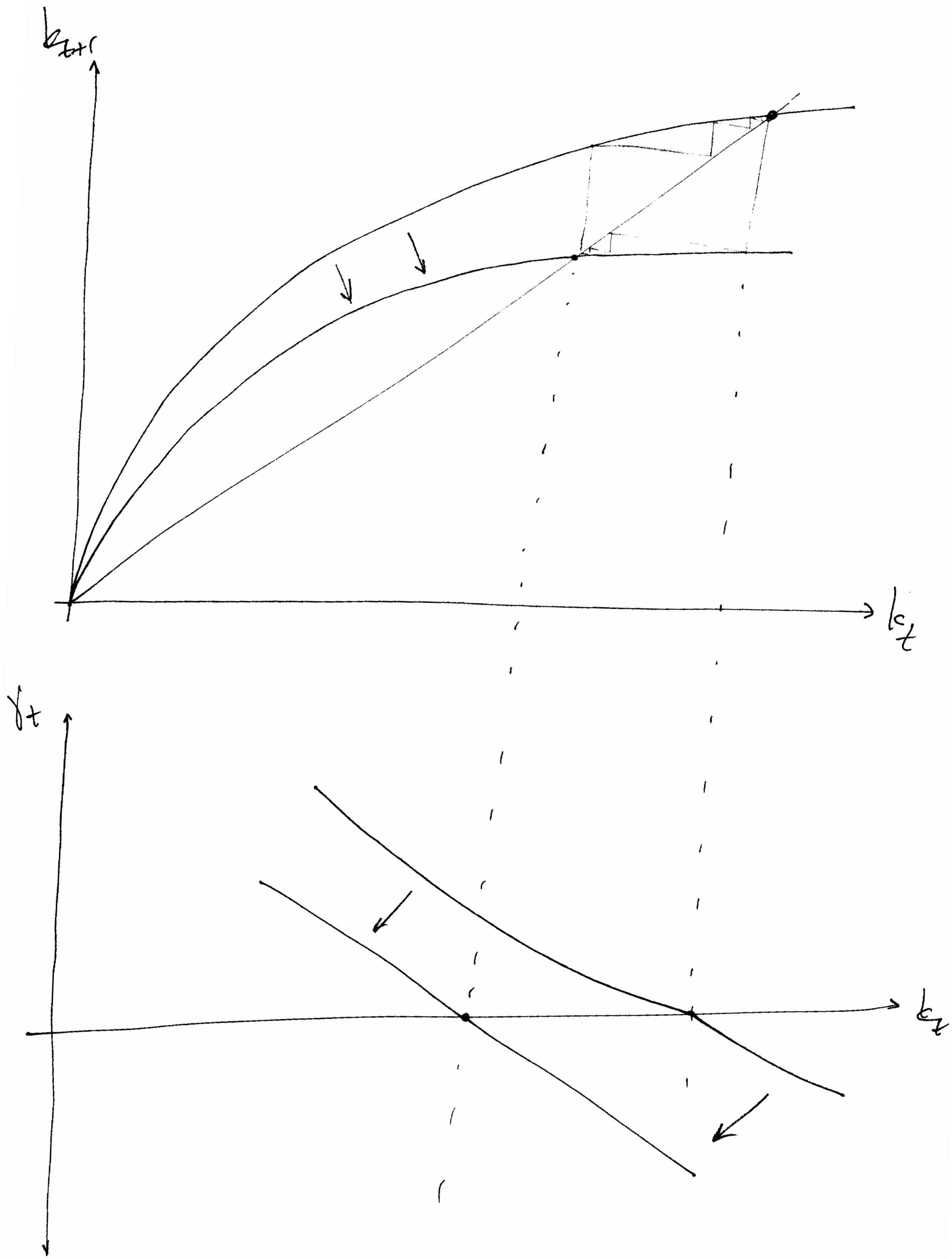

FIGURE 4

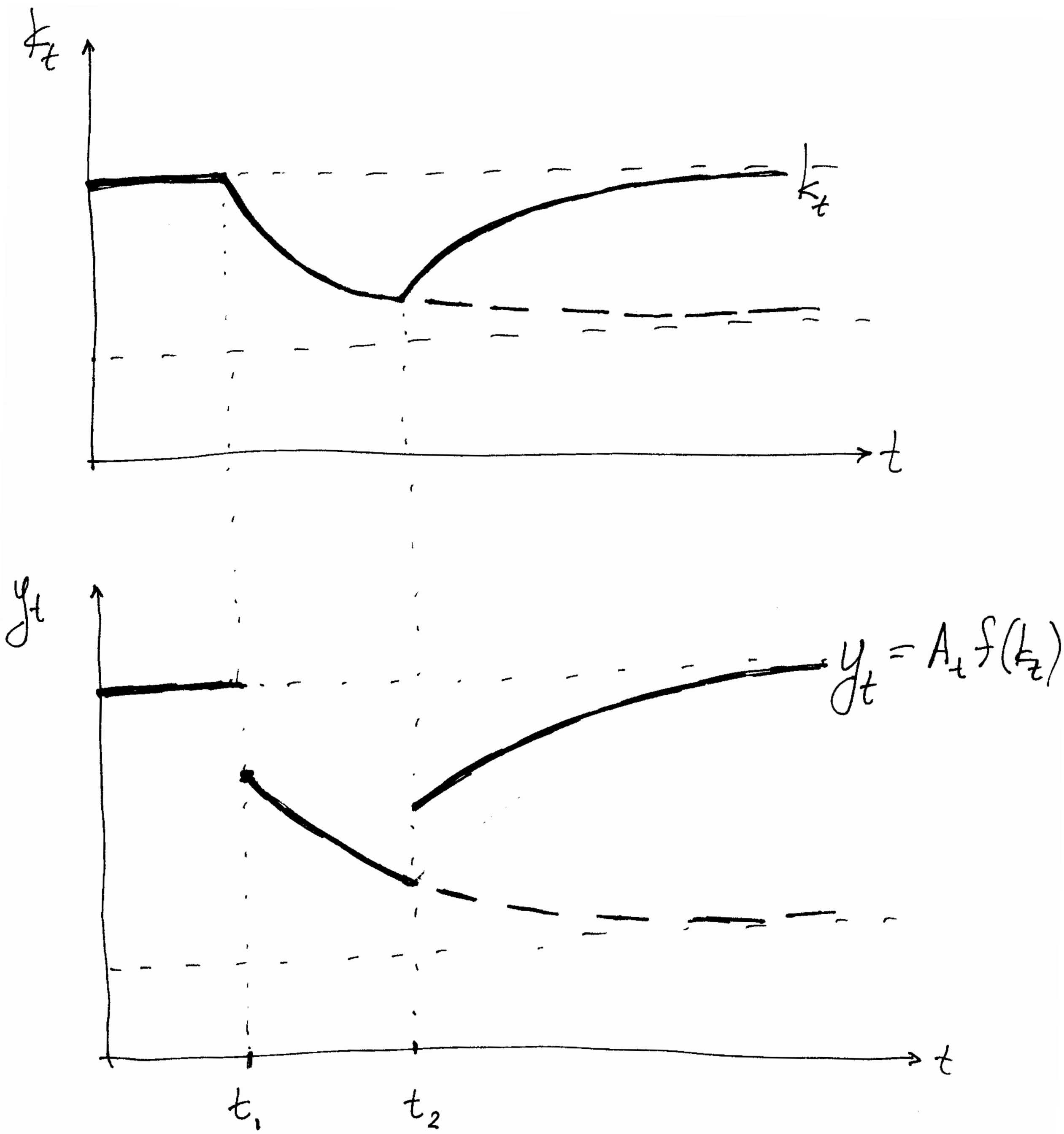

FIGURE 5

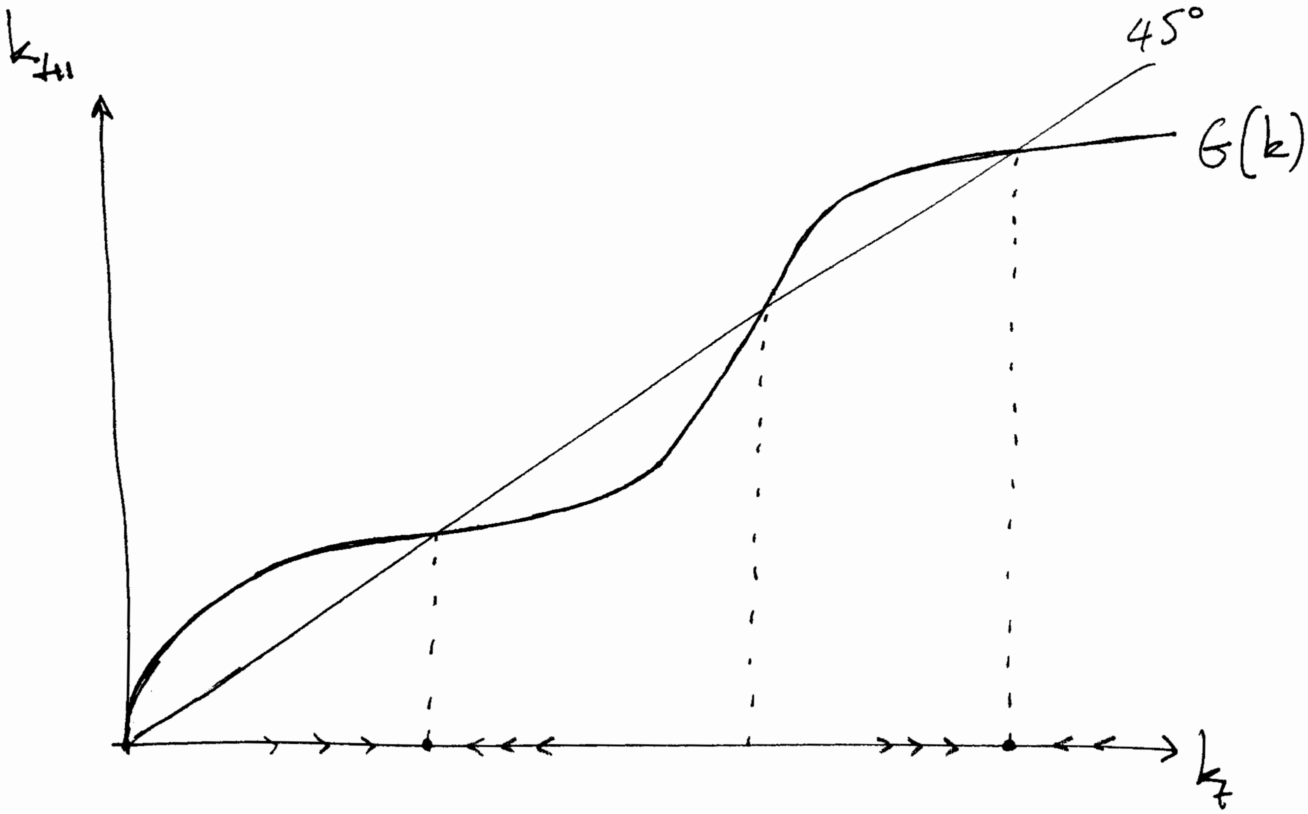

FIGURE 6

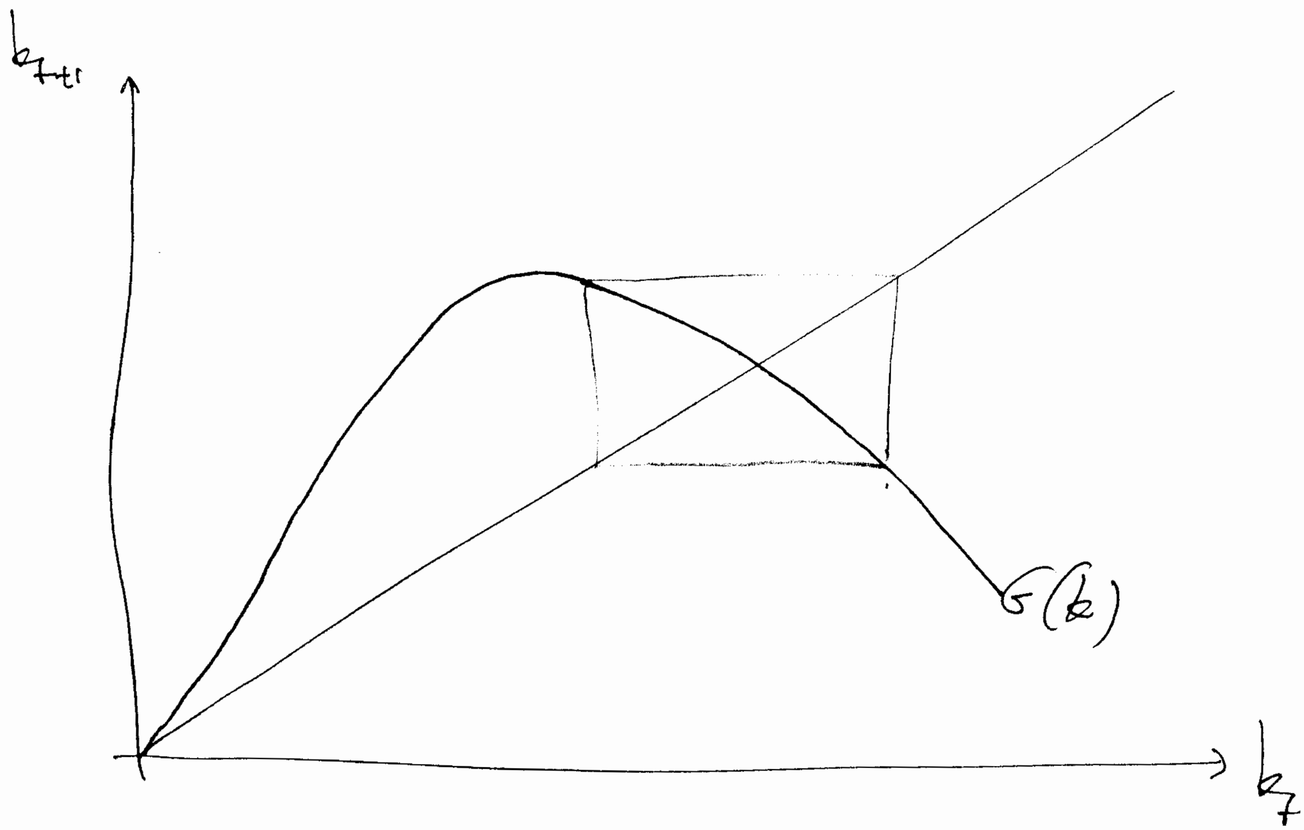

FIGURE 7