14.05 Lecture Notes

Endogenous Growth

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1   The Simple AK Model

- In this section we consider the simplest version of a model with endogenous long-run growth

- The key objective here is to understand how long-run growth emerges once we take a broad view of what "capital" means, and allow for non-vanishing returns to capital accumulation

- Total output in the economy is given by

$$Y_t = AK_t,$$

  where $A > 0$ is an exogenous parameter.

- In intensive (per-capita) form,

$$y_t = f(k_t) = Ak_t.$$

1.1 Pareto Allocations

- The social planner's problem is the same as in the Ramsey model, except for the fact that output is linear in capital:

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$s.t. \quad c_t + k_{t+1} \leq f(k_t) + (1 - \delta)k_t$$

- The Euler condition gives

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta \left(1 + A - \delta\right)$$

- Assuming CEIS, this reduces to

$$\frac{c_{t+1}}{c_t} = [\beta\,(1 + A - \delta)]^{\theta}$$

  or

$$\ln c_{t+1} - \ln c_t \approx \theta(A - \delta - \rho)$$

  That is, consumption growth is proportional to the difference between the real return on capital and the discount rate. Note that now the real return is a constant, rather than diminishing with capital accumulation.

- The resource constraint can be rewritten as

$$c_t + k_{t+1} = (1 + A - \delta)k_t.$$

Since total resources (the RHS of the above) are linear in k and preferences are homothetic, an educated guess is that optimal consumption and investment are also linear in k. We thus propose

$$c_t = (1 - s)(1 + A - \delta)k_t$$
$$k_{t+1} = s(1 + A - \delta)k_t$$

where the coefficient s is to be determined and must satisfy $s \in (0, 1)$ for the solution to exist.

- It follows that

$$\frac{c_{t+1}}{c_t} = \frac{k_{t+1}}{k_t} = \frac{y_{t+1}}{y_t}$$

so that consumption, capital and income all grow at the same rate.

- To ensure perpetual growth, we thus need to impose

$$\beta\left(1 + A - \delta\right) > 1,$$

or equivalently $A - \delta > \rho$. If that condition were not satisfied, and instead $A - \delta < \rho$, then the economy would shrink at a constant rate towards zero.

- From the resource constraint we then have

$$\frac{c_t}{k_t} + \frac{k_{t+1}}{k_t} = (1 + A - \delta),$$

implying that the consumption-capital ratio is given by

$$\frac{c_t}{k_t} = (1 + A - \delta) - [\beta\,(1 + A - \delta)]^{\theta}$$

Using $c_t = (1 - s)(1 + A - \delta)k_t$ and solving for s we conclude that the optimal saving rate is

$$s = \beta^{\theta}\,(1 + A - \delta)^{\theta - 1}.$$

Equivalently, $s = \beta^{\theta}(1 + R)^{\theta - 1}$, where $R = A - \delta$ is the net social return on capital. Note that the saving rate is increasing (decreasing) in the real return if and only if the EIS is higher (lower) than unit, and $s = \beta$ for $\theta = 1$.

- Finally, to ensure $s \in (0, 1)$, we impose

$$\beta^\theta \left(1 + A - \delta\right)^{\theta-1} < 1.$$

This is automatically ensured when $\theta \leq 1$ and $\beta \left(1 + A - \delta\right) > 1$, as then $s = \beta^\theta \left(1 + A - \delta\right)^{\theta-1} \leq \beta < 1$. But when $\theta > 1$, this puts an upper bound on A. If A exceeded that upper bound, then the social planner could attain infinite utility, and the problem is not well-defined.

1.2 The Competitive Economy

- There is a large number of competitive firms, each with access to the same AK technology. The equilibrium rental rate of capital and the equilibrium wage rate are then simply given by

$$r = A \quad \text{and} \quad w = 0.$$

  while arbitrage between bonds and capital implies that the interest rate is $R = r - \delta = A - \delta$.

- On the other hand, the Euler condition for the representative household gives

$$\frac{c_{t+1}}{c_t} = [\beta (1 + R)]^\theta.$$

- Following the same steps as in the neoclassical growth model, we conclude that the competitive market allocations coincide with the Pareto optimal ones. This is true only because the private and the social return to capital coincide.

Proposition 1 *Consider an AK economy with CEIS preferences and suppose that* $(\beta, \theta, A, \delta)$ *satisfy* $\beta \left(1 + A - \delta\right) > 1 > \beta^{\theta} \left(1 + A - \delta\right)^{\theta - 1}$. *Then, the equilibrium is pareto efficient and the economy exhibits a balanced growth path. Capital, output, and consumption all grow at a constant rate, which is given by*

$$\frac{k_{t+1}}{k_t} = \frac{y_{t+1}}{y_t} = \frac{c_{t+1}}{c_t} = \left[\beta \left(1 + A - \delta\right)\right]^{\theta} > 1.$$

Consumption and investment are given by

$$c_t = \left(1 - s\right) \left(1 + A - \delta\right) k_t \quad and \quad k_{t+1} = s \left(1 + A - \delta\right) k_t.$$

where

$$s = \beta^{\theta} \left(1 + A - \delta\right)^{\theta - 1}.$$

- The growth rate is increasing in A, θ, and β. Differences in productivity or preferences may thus explain differences, not only in the level of output and the rate of investment, but also in rate of long-run growth.

1.3 Beyond AK

- The analysis here has assumed a single type of capital and a single sector of production. We could consider multiple types of capital and multiple sectors. In essence, we can "endogenize" the capital K and the productivity A—for example, in terms of physical versus human capital, intentional capital accumulation versus unintentional spillovers, innovation and knowledge creation, etc. The level of productivity and the growth rate then depend on how the economy allocates resources across different types of capital and different sectors of production.

- In the textbook, you can find a version of the Solow/Ramsey model with two types of capital, physical (or tangible) and human (or intangible). If the production functions is homogenous of degree one in the pair of these two types of capital, then the economy is effectively an AK economy, in the sense that as we double both types of capital we also double output. The long-run growth behavior of this economy is then essentially the same as the one we have considered here. However, the important property to remember is that, because markets are competitive and there are no externalities, Pareto and competitive allocations coincide.

- In the next sections, we consider richer endogenous growth models that achieve two goals:

  - provide a deeper understanding of the process of technological growth (positive), shifting focus in particular to the accumulation of new knowledge

  - open up the door to inefficiency in market allocations (normative), and hence to interesting policy implications

2 Learning by Doing and Spillovers

2.1 Market Allocations

- Output for firm m is given by

$$Y_t^m = F\left(K_t^m, h_t L_t^m\right)$$

  where h_t represents the aggregate level of human capital or knowledge.

- h_t is endogenously determined in the economy (as we will specify in a moment how) but it is exogenous from the perspective of either firms or households.

- Firm profits are given by

$$\Pi_t^m = F\left(K_t^m, h_t L_t^m\right) - r_t K_t^m - w_t L_t^m$$

The FOCs give

$$
\begin{aligned}
r_t &= F_K\left(K_t^m, h_t L_t^m\right) \\
w_t &= F_L\left(K_t^m, h_t L_t^m\right) h_t
\end{aligned}
$$

Using the market clearing conditions for physical capital and labor, we infer $K_t^m/L_t^m = k_t$, where k_t is the aggregate capital labor ratio in the economy. We conclude that, given k_t and h_t, market prices are given by

$$
\begin{aligned}
r_t &= F_K(k_t, h_t) = f'(\kappa_t) \\
w_t &= F_L(k_t, h_t)h_t = [f(\kappa_t) - f'(\kappa_t)\kappa_t]h_t
\end{aligned}
$$

where $f(\kappa) \equiv F(\kappa, 1)$ is the production function in intensive form and $\kappa_t = k_t/h_t$.

- Households, like firms, take w_t, r_t and h_t as exogenous to their decisions. The representative household maximizes utility subject to the budget constraint

$$c_t + k_{t+1} + b_{t+1} \leq w_t + (1 + r_t - \delta)k_t + (1 + R_t)b_t.$$

Arbitrage between bonds and capital imply $R_t = r_t - \delta$, while the Euler condition gives

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta\,(1 + R_t) = \beta(1 + r_t - \delta).$$

- To close the model, we need to specify how h_t is determined. Following Arrow and Romer, we assume that knowledge accumulation is the unintentional by-product of learning-by-doing in production. We thus let the level of knowledge to be proportional to either the level of output, or the level of capital:

$$h_t = \eta k_t,$$

for some constant $\eta > 0$.

- It follows that the ratio $k_t/h_t = \kappa_t$ is pinned down by

$$\kappa_t = 1/\eta.$$

Letting the constants A and ω be defined

$$A \equiv f'(1/\eta) \quad \text{and} \quad \omega \equiv f(1/\eta)\eta - f'(1/\eta),$$

we infer that equilibrium prices are given by

$$r_t = A \quad \text{and} \quad w_t = \omega k_t.$$

Substituting into the Euler condition gives

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta \left(1 + A - \delta\right).$$

- Finally, it is immediate that capital and output grow at the same rate as consumption.

Proposition 2 *Let $A \equiv f'(1/\eta)$ and $\omega \equiv f(1/\eta)\eta - f'(1/\eta)$, and suppose $\beta\left(1 + A - \delta\right) > 1 > \beta^\theta\left(1 + A - \delta\right)^{\theta-1}$. Then, the market economy exhibits a balanced growth path. Physical capital, knowledge, output, wages, and consumption all grow at a constant rate given by*

$$\frac{y_{t+1}}{y_t} = \frac{c_{t+1}}{c_t} = \left[\beta\left(1 + A - \delta\right)\right]^\theta > 1.$$

The wage rate is given by $w_t = \omega k_t$, while the investment rate out of total resources is given by $s = \beta^\theta\left(1 + A - \delta\right)^{\theta-1}$.

2.2 Pareto Allocations and Policy Implications

- Consider now the Pareto optimal allocations. The social planner recognizes that knowledge in the economy is proportional to physical capital and internalizes the effect of learning-by-doing. He thus understands that output is given by

$$y_t = F(k_t, h_t) = A^* k_t$$

where

$$A^* \equiv A + \omega = f(1/\eta)\eta$$

represents the *social* return on capital. It is therefore as if the social planner had access to a linear technology like in the simple Ak model, and therefore the Euler condition for the social planner is given by

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta \left(1 + A^* - \delta\right).$$

- The social return to capital is higher than the private (market) return to capital:

$$A^* > A = r_t$$

The difference is ω, the fraction of the social return on saving that is "wasted" as labor income.

Proposition 3 *Let $A^* \equiv A + \omega \equiv f(1/\eta)\eta$ and suppose $\beta(1 + A^* - \delta) > 1 > \beta^\theta (1 + A^* - \delta)^{\theta-1}$.
Then, the Pareto optimal plan exhibits a balanced growth path. Physical capital, knowledge, output,
wages, and consumption all grow at a constant rate given by*

$$\frac{y_{t+1}}{y_t} = \frac{c_{t+1}}{c_t} = [\beta(1 + A^* - \delta)]^\theta > 1.$$

Note that $A < A^*$, and therefore the market growth rate is lower than the Pareto optimal one.

- *Exercise:* Reconsider the market allocation and suppose the government intervenes by subsidizing either private savings or firm investment. Find, in each case, what is the subsidy that implements the optimal growth rate. Is this subsidy the optimal one, in the sense that it maximizes social welfare?

3 R&D and Innovation

- The economy is populated by a large number of "entrepreneurs". Each entrepreneur lives (is present in the market) for $1 + T$ periods, where T is random. Conditional on being alive in the present period, there is probability n that the entrepreneur will die (exit the market) by the end of that period. n is constant over time and independent of age. In each period, a mass n of existing entrepreneurs dies, and a mass n of new entrepreneurs is born, so that the population is constant.

- In the first period of life, the entrepreneur is endowed with the aggregate level of knowledge in the economy. In the first period of life, he also has a "fresh mind" and can engage in R&D activity. In later periods of life, instead, he is too old for coming up with good new ideas and therefore engages only in production, not innovation.

- Young producers engage in R&D in order to increase the profits of their own productive activities later in life. But individual innovation has spillover effects to the whole economy. When a mass of producers generate new ideas, the aggregate level of knowledge in the economy increases proportionally with the production of new ideas.

3.1 R&D Technology

- Let V_{t+1}^j denote the value of an innovation for individual j realized in period t and implemented in period $t+1$. Let z_t^j denote the amount of skilled labor that a potential innovator j employes in R&D and $q(z_t^j)$ the probability that such R&D activity will be successful. $q : \mathbb{R} \to [0,1]$ represents the technology of producing innovations and satisfies $q(0) = 0$, $q' > 0 > q''$, $q'(0) = \infty$, $q'(\infty) = 0$.

- The potential researcher maximizes

$$q(z_t^j) \cdot V_{t+1}^j - w_t \cdot z_t^j.$$

It follows that the optimal level of R&D is given by $q'(z_t^j)V_{t+1}^j = w_t$ or

$$z_t^j = g\left(V_{t+1}^j/w_t\right)$$

where the function $g(v) \equiv (q')^{-1}(1/v)$ satisfies $g(0) = 0$, $g' > 0$, $g(\infty) = \infty$. Note that z will be stationary only if both V and w grow at the same rate.

3.2   The Value of Innovation

- What determines the value of an innovation? For a start, let us assume a very simple structure. Let A_t^j represent the TFP of producer j in period t. The profits from his production are given by

$$\Pi_t^j = A_t^j \widehat{\pi}$$

where $\widehat{\pi}$ represents normalized profits. We can endogenize π, but we won't do it here for simplicity.

- When a producer is born, he automatically learns what is the contemporaneous aggregate level of technology. That is, $A_t^j = A_t$ for any producer born in period t. In the first period of life, and only in that period, a producer has the option to engage in R&D. If his R&D activity fails to produce an innovation, them his TFP remains the same for the rest of his life. If instead his R&D activity is successful, then his TFP increases permanently by a factor $1 + \gamma$, for some $\gamma > 0$.

- That is, for any producer j born in period t, and for all periods $\tau \geq t+1$ in which he is alive,

$$
A_\tau^j = \begin{cases} A_t & \text{if his R\&D fails} \\ (1+\gamma)A_t & \text{if his R\&D succeeds} \end{cases}
$$

- It follows that a successful innovation generates a stream of "dividends" equal to $\gamma A_t \widehat{\pi}$ per period for all $\tau > t$ that the producer is alive. Therefore,

$$
V_{t+1} = \sum_{\tau=t+1}^{\infty} \left(\frac{1-n}{1+R} \right)^\tau (\gamma A_t \widehat{\pi}) = \gamma \widehat{v} A_t \tag{1}
$$

where where R is the interest rate per period and

$$
\widehat{v} \equiv \sum_{\tau=1}^{\infty} \left(\frac{1-n}{1+R} \right)^\tau \widehat{\pi} \approx \frac{\widehat{\pi}}{R+n}.
$$

Note that the above would be an exact equality if time was continuous. Note also that $\widehat{v}$ is decreasing in both R and n.

- *Remark:* We see that the probability of "death" reduces the value of innovation, simply because it reduces the expected life of the innovation. Here we have taken n as exogenous

for the economy. But later we will endogenize n. We will recognize that the probability of "death" simply the probability that the producer will be displaced by another competitor who manages to innovate and produce a better substitute product. For the time being, however, we treat n as exogenous.

3.3 The Cost of Innovation

- Suppose that skilled labor has an alternative employment, which a simple linear technology of producing final goods at the current level of aggregate TFP. That is, if l_t labor is used in production of final goods, output is given by $A_t l_t$. Since the cost of labor is w_t, in equilibrium it must be that

$$w_t = A_t. \tag{2}$$

3.4   Equilibrium

- Combining (1) and (2), we infer that

$$\frac{V_{t+1}}{w_t} = \gamma \widehat{v}$$

  It follows that the level of R&D activity is the same across all new-born producers:

$$z_t^j = z_t = g\left(\gamma \widehat{v}\right).$$

- The outcome of the R&D activity is stochastic for the individual. By the LLN, however, the aggregate outcome is deterministic. The aggregate rate of innovation is simply

$$\lambda_t = q(z_t) = \lambda\left(\gamma \widehat{v}\right)$$

  where $\lambda\left(x\right) \equiv q\left(g\left(x\right)\right).$

- It follows that the aggregate level of technology grows at a rate

$$\frac{A_{t+1}}{A_t} = 1 + \gamma \lambda_t = 1 + \gamma \lambda\left(\gamma \widehat{v}\right).$$

- An increase in $\widehat{\pi}$ increases the incentives for R&D in the individual level and therefore results to higher rates of innovation and growth in the aggregate level. An increase in γ has a double effect. Not only it increases the incentive for R&D, but it also increase the spill over effect from individual innovations to the aggregate level of technology.

- How does an increase in R or n affect the rates of innovation and growth?

- What is aggregate output in the economy? It's the sum of the output of all entrepreneurs plus the output of workers not employed in $R\&D$. Check that aggregate output grows at the same rate as aggregate knowledge.

3.5 Business Stealing

- Consider a particular market j, in which a producer j has monopoly power. Suppose now that there is an outside competitor who has the option to engage in R&D in an attempt to create a better product that is a close substitute for the product of producer j. Suppose further that, if successful, the innovation will be so "radical" that, not only it will increase productivity and reduce production costs, but it will also permit the outsider to totally displace the incumbent from the market.

- *Remark:* Here we start seeing how both production and innovation may depend on the IO structure. In more general versions of the model, the size of the innovation and the type of competition (e.g., Bertrand versus Cournot) determine what is the fraction of monopoly profits that the entrant can grasp and hence the private incentives for innovation.

- What is the value of the innovation for this outsider? Being an outsider, he has no share in the market of product j. If his R&D is successful, he expects to displace the incumbent and grasp the whole market of product j. That is, an innovation delivers a dividend equal to total market profits, $(1 + \gamma)A_t\widehat{\pi}$, in each period of life. Assuming that the outsider also has a probability of death (or displacement) equal to n, the value of innovation for the outsider is given by

$$V_{t+1}^{out} = \sum_{\tau=t+1}^{\infty} \left(\frac{1-n}{1+R}\right)^{\tau} [(1+\gamma)A_t\widehat{\pi}] = (1+\gamma)\widehat{v}A_t$$

- Now suppose that the incumbent also has the option to innovate is later periods of life. If he does so, he will learn the contemporaneous aggregate level of productivity and improve upon it by a factor $1 + \gamma$. The value of innovation in later periods of life is thus the same as in the first period of life:

$$V_{t+1}^{in} = \sum_{\tau=t+1}^{\infty} \left(\frac{1-n}{1+R}\right)^{\tau} [\gamma A_t\widehat{\pi}] = \gamma\widehat{v}A_t.$$

- Obviously, $V_{t+1}^{out} > V_{t+1}^{in}$. This is because the incumbent values only the potential increase in productivity and profits, while the outsider values in addition the profits of the incumbent. This "business-stealing" effect implies that, ceteris paribus, that innovation will originate mostly in outsiders.

- *Remark:* In the standard Aghion-Howitt model, as opposed to the variant considered here, *only* outsiders engage in innovation. Think why this is the case in that model, and why this might not be the case here. Then, find conditions on the technology q and the parameters of the economy that would ensure in our model a corner solution for the insiders and an interior solution for the outsiders. (Hint: you may need to relax the Inada condition for q.) We will henceforth assume that only outsiders engage in innovation.

- *Remark:* Things could be different if the incumbent has a strong cost advantage in R&D, which could be the case if the incumbent has some private information about the either the technology of the product or the demand of the market.

- Assuming that only outsiders engage in R&D, and using $V_{t+1}^{out}/w_t = (1 + \gamma)\widehat{v}$, we infer that the optimal level of R&D for an outsider is

$$z_t^{out} = z_t = g\left((1 + \gamma)\widehat{v}\right).$$

and therefore the aggregate rate of innovation is

$$\lambda_t = q(z_t) = \lambda\left((1 + \gamma)\widehat{v}\right)$$

We conclude that the growth rate of the economy is

$$\frac{y_{t+1}}{y_t} = \frac{A_{t+1}}{A_t} = 1 + \gamma\lambda\left((1 + \gamma)\widehat{v}\right).$$

- We can now reinterpret the probability of "death" as simply the probability of being displaced by a successful outside innovator. Under this interpretation, we have

$$n = \lambda \left((1 + \gamma)\widehat{v} \right)$$

and $\widehat{v}$ solves

$$\widehat{v} = \frac{\widehat{\pi}}{R + \lambda \left((1 + \gamma)\widehat{v} \right)}$$

- Note that an increase in $\widehat{\pi}$ will now increase $\widehat{v}$ by less than one-to-one, because the displacement rate will also increase.

3.6 Efficiency and Policy Implications

- Discuss the spillover effects of innovation... Both negative and positive...

- Discuss optimal patent protection... Trade-off between incentives and externalities...

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