1. (27 points) For each of the following production functions, sketch a representative isoquant (2 points). Calculate the marginal product for each input, and indicate whether each marginal product is diminishing, constant, or increasing (3 points). Also calculate the marginal rate of technical substitution for each function (2 points). Also indicate whether the function exhibits constant, increasing, or diminishing returns to scale (2 points).

   (a) $F(L, K) = LK^3$

   This production function is of the Cobb-Douglas form. Isoquants: $K = (Q/L)^{\frac{1}{3}}$ look approximately as follows:
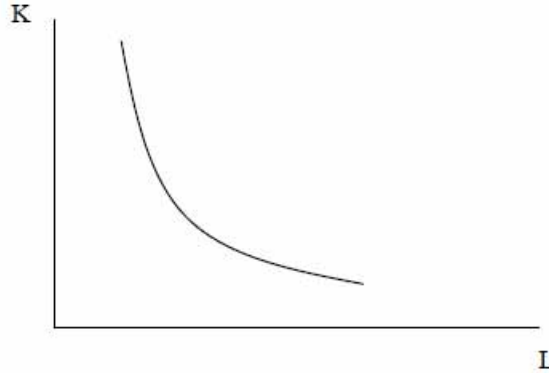


   Let's first calculate the marginal products and check whether they are diminishing, constant, or increasing.

$$MP_L = \frac{\partial F}{\partial L} = K^3 \qquad\qquad \frac{\partial^2 F}{\partial L^2} = 0 \qquad\qquad \rightarrow \text{constant}$$

$$MP_K = \frac{\partial F}{\partial K} = 3LK^2 \qquad\qquad \frac{\partial^2 F}{\partial L^2} = 6LK > 0 \qquad\qquad \rightarrow \text{increasing}$$

   For the MRTS we get $MRTS = \frac{MP_L}{MP_K} = \frac{K^3}{3LK^2} = \frac{K}{3L}$. Checking for returns to scale (we are scaling up all inputs for a factor $t > 1$):

$$F(tK, tL) = tL(tK)^3 = t^4 LK^3 = t^4 \cdot F(K, L)$$

   So, this production function exhibits increasing returns to scale.

   (b) $F(L, K) = L + 3K$

   This is the perfect substitutes production function. Isoquants are linear with a slope equal to the negative of the constant MRTS, which is 1/3 here. Isoquant: $K = Q/3 - L/3$
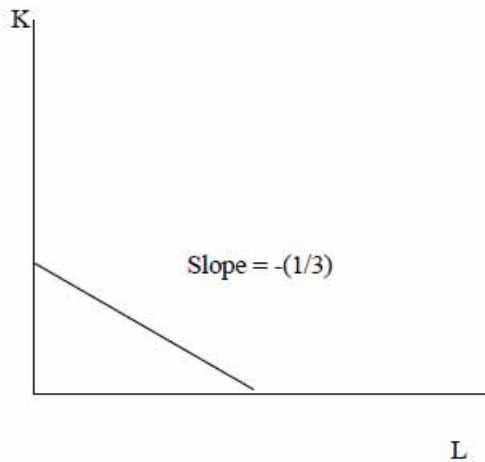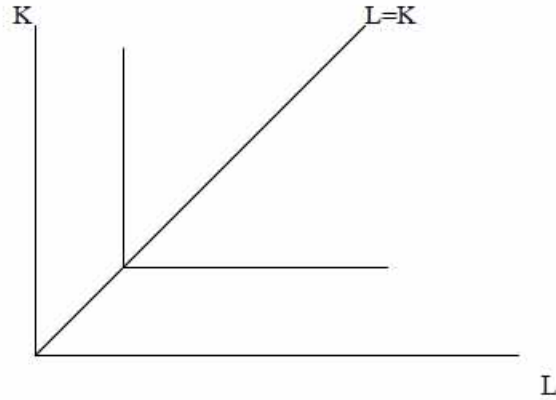
Marginal products are constant for both inputs.

$$MP_L = \frac{\partial F}{\partial L} = 1 \qquad\qquad \frac{\partial^2 F}{\partial L^2} = 0 \qquad\qquad \rightarrow \text{constant}$$

$$MP_K = \frac{\partial F}{\partial K} = 3 \qquad\qquad \frac{\partial^2 F}{\partial L^2} = 0 \qquad\qquad \rightarrow \text{constant}$$

For the MRTS we get $MRTS = \frac{MP_L}{MP_K} = \frac{1}{3}$. We verify constant returns to scale:

$$F(tK, tL) = tL + 3tK = t(L + 3K) = t \cdot F(K, L)$$

(c) $F(L, K) = (\min\{L, K\})^{\frac{1}{3}}$

This is a perfect complements production function. Isoquants are L-shaped, with a kink at $L = K$.



Marginal products for both inputs:

$$MP_L = \frac{\partial F}{\partial L} = \begin{cases} 0 & \text{if } L \geq K \\ \frac{1}{3}L^{-\frac{2}{3}} & \text{if } L < K \end{cases} \qquad\qquad \frac{\partial^2 F}{\partial L^2} = \begin{cases} 0 & \text{if } L \geq K \\ -\frac{2}{9}L^{-\frac{5}{3}} & \text{if } L < K \end{cases}$$

$$MP_K = \frac{\partial F}{\partial K} = \begin{cases} 0 & \text{if } L \leq K \\ \frac{1}{3}K^{-\frac{2}{3}} & \text{if } L > K \end{cases} \qquad\qquad \frac{\partial^2 F}{\partial L^2} = \begin{cases} 0 & \text{if } L \leq K \\ -\frac{2}{9}K^{-\frac{5}{3}} & \text{if } L > K \end{cases}$$

For the MRTS we get $MRTS = \frac{MP_L}{MP_K} = \begin{cases} 0 & \text{if } L > K \\ \infty & \text{if } L < K \end{cases}$

MRTS is not defined on the kink points, where $K = L$. This production function exhibits decreasing returns to scale (note $t > 1$):

$$F(tL, tK) = [\min\{tL, tK\}]^{\frac{1}{3}} = [t \cdot \min\{L, K\}]^{\frac{1}{3}} = t^{\frac{1}{3}}[\min\{L, K\}]^{\frac{1}{3}} < t \cdot F(K, L)$$

2. (21 points) Consider the production function $f(L, K) = 2L^{\frac{1}{4}}K^{\frac{1}{4}}$.

(a) (15 points) Find the associated (long run) total, average, and marginal cost curves.

We want to minimize $wL + rK$ so that $2L^{\frac{1}{4}}K^{\frac{1}{4}} = Q$. Setting up the Lagrangian and solving for the first order conditions yields the "conditional" input demand curves for K and L.

$$L^*(r, w, Q) = \frac{Q^2 r^{\frac{1}{2}}}{4w^{\frac{1}{2}}}, \quad K^*(r, w, Q) = \frac{Q^2 w^{\frac{1}{2}}}{4r^{\frac{1}{2}}}$$

$$TC^*(r, w, Q) = \frac{Q^2 w^{\frac{1}{2}} r^{\frac{1}{2}}}{2}$$

$$AC^*(r, w, Q) = \frac{Q^2 w^{\frac{1}{2}} r^{\frac{1}{2}}}{2Q} = \frac{Q w^{\frac{1}{2}} r^{\frac{1}{2}}}{2}$$

$$MC^*(r, w, Q) = \frac{\partial TC}{\partial Q} = \frac{2Q w^{\frac{1}{2}} r^{\frac{1}{2}}}{2} = Q w^{\frac{1}{2}} r^{\frac{1}{2}}$$

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(b) (6 points) Sketch the total, average, and marginal cost curves.
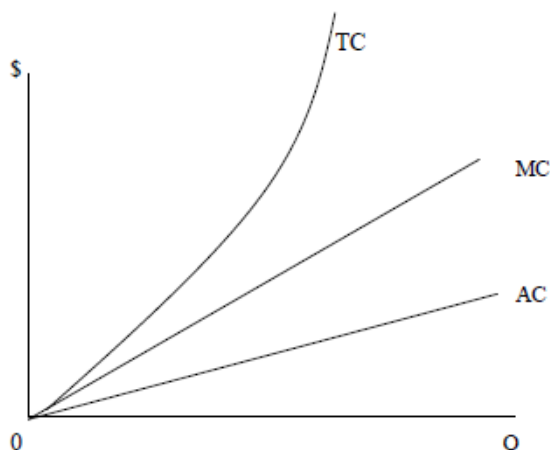


3. Problem solution removed due to copyright restrictions. This content is presented in audio form in the Solution Video for Problem Set 4, Problem 3.

4. (28 points) You run a cost-minimizing firm with production function $f(L, K) = [\min\{L, K\}]^{\frac{1}{3}}$, where $L$ is labor and $K$ is capital. Assume that you are a price-taker in the input markets: you pay $w$ for each unit of labor you hire and $r$ for each unit of capital (where $w$ and $r$ are set exogenously), and face no costs other than those from labor and capital.

   (a) (15 points) Assuming that you can freely choose both labor and capital (i.e., the "long run problem"), derive expressions for your cost-minimizing conditional input demands, $L^*(r, w, Q)$ and $K^*(r, w, Q)$. Confirm that the conditional input demand functions are "homogeneous of degree zero" in $w$ and $r$; that is,

   $$\begin{cases} L^*(tr, tw, Q) = L^*(r, w, Q) \\ K^*(tr, tw, Q) = K^*(r, w, Q) \end{cases} \quad \text{for all } t > 0$$

   $F(L, K) = \{\min L, K\}^{\frac{1}{3}}$ is a perfect complements production function, so we know that we will always be producing at $L = K$. Using this, $Q = F(L, K) = \{\min L, K\}^{\frac{1}{3}} = \{\min L, L\}^{\frac{1}{3}} = L^{\frac{1}{3}} \longrightarrow$ $L^* = Q^3$ and $K^* = Q^3$.

   That is, $L^*(r, w, Q) = K^*(r, w, Q) = Q^3$ are independent of the input prices $r$ and $w$. In particular, we verify homogeneity of degree zero: $L^*(tr, tw, Q) = Q^3 = L^*(r, w, Q)$ and same for $K^*(r, w, Q)$.

   (b) (8 points) What will happen to your conditional demand for labor if there is an increase in the wage rate, assuming that $r$ and $Q$ remain the same? Explain in one sentence why your answer makes intuitive sense.

   As can be seen both in the picture and by plugging into the formulas, since $L^*(r, w, Q) = K^*(r, w, Q) = Q^3$ are independent of the prices $r$ and $w$, there will be no change in the inputs for a change in prices (given the perfect complement production function, producers cannot substitute away from labor when its price $w$ increases).

   (c) (5 points) Use your answers from (a) to write down an expression for your total cost function $TC(r, w, Q)$. Is this function "homogeneous of degree one" in $w$ and $r$; that is, does $TC(tr, tw, Q) = t \cdot TC(r, w, Q)$?

   The total cost function $TC(r, w, Q)$ states how much it costs to produce quantity $Q$, when prices are $r$ and $w$ (and the producer behaves optimally). We have:

   $$TC(r, w, Q) = wL^*(r, w, Q) + rK^*(r, w, Q) = wQ^3 + rQ^3 = (r + w)Q^3$$

   $TC(r, w, Q)$ is indeed homogenous of degree one:

   $$TC(tr, tw, Q) = (tr + tw)Q^3 = t \cdot (r + w)Q^3 = t \cdot TC(r, w, Q)$$

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