Chapter 20

Boundary layer turbulence

Turbulence in the ocean and atmosphere is strongly affected by the presence of boundaries. Boundaries impose severe modifications to the momentum and buoyancy budgets. At solid boundaries, the boundary condition that the fluid velocity is zero applies to both the mean velocity and to the fluctuations. Thus the turbulent fluxes of momentum must vanish. At the ocean free surface winds apply a stress that drives strongly turbulent motions. Surface fluxes of heat, salt and moisture can generate turbulent convection as we have seen at the beginning of this class. Finally the combination of surface stresses and fluxes determines the full spectrum of turbulent motions that can develop at the ocean and atmosphere boundaries. Before discussing in detail the physics of planetary boundary layers in the ocean and atmosphere, it is useful to review some fundamental results that apply to boundary turbulence in general.

20.1 Shear boundary layers

Let us consider shear-driven turbulence at solid boundaries. At flluid boundaries, the condition that the fluid velocity is zero applies at every instant in time. Thus it applies to the mean velocity and the fluctuations separately,

$$\bar{\boldsymbol{u}} = 0, \qquad \boldsymbol{u}' = 0. \tag{20.1}$$

The fact that the fluctuations drop to zero at the wall has the particular implication that the Reynolds stress vanish,

$$-\overline{u_i u_j} = 0. \tag{20.2}$$

The only stress exerted directly on the wall is the viscous one. Away from the wall, instead, turbulence generates a Reynolds stress typically large compared to the viscous stress. Tritton (chapter 5, page 337) shows in Figure 21.12 the transition between a viscous stress and a turbulent stress in a turbulent boundary layer experiment (Schubauer, J. Appl. Physics, 1954). The total stress parallel to the wall does not change with distance from the wall, but there is an exchange of balance between the viscous and turbulent contributions.

Further reading: Tritton, chapter 21, 336–344

To simplify the algebra let us consider a parallel irrotational flow over a flat boundary. Turbulence is generated because the no-slip condition $\bar{u} = 0$ at the boundary means that a shear layer results, and vorticity is introduced into the flow. Boundary-layer flows are more complicated than free shear flows, because the importance of viscosity at the boundaries (which enforces the no-slip condition) introduces a new spatial scale in the problem. As a result there is a viscous sublayer next to the wall, whose width is set by viscous forces, and a high Re boundary layer, whose thickness is controlled by the turbulent Reynolds stresses. These two layers are separated by an inertial sublayer. The three different regions of the boundary layer are somewhat analogous to the viscous range, inertial range, and forcing ranges of isotropic, homogeneous turbulence.

1. The viscous sublayer
For distances close to the wall, i.e. $z < z_f$ where z_f is the distance at which $Re = 1$, friction is important. This can be compared to length scales $l \approx 1/k_d$ in homogeneous turbulence, where viscosity is important.

2. The inertial sublayer
At distances further away from the wall than z_f, we can neglect viscosity. Similarly, if we are not close to the edge of the boundary layer at $z = \delta$, we can assume that the flow will not depend directly on the size of the boundary layer. Therefore we have an inertial sublayer for $z_f << z << \delta$. This region is similar to the inertial range in homogeneous turbulence, where the flow is not affected by ν or by k_0, the wavenumber of the energy input.

3. The turbulent boundary layer
The full turbulent boundary layer is determined by the maximum size of the eddies, the so-called the integral scale δ. This region corresponds to the forcing range of 3D turbulence.

4. The ambient flow
Finally at some distance $z > \delta$, the flow is no longer turbulent and we are in the irrotational ambient flow.

Further reading: Tennekes and Lumley, chapter 5, 147–163.

20.1.1 Equations of motion

We will assume a constant background flow $\bar{u}_0$, which is independent of distance along the plate x and distance normal to the plate z. We assume 2-dimensional flow $(\partial/\partial y = 0)$, and also assume that downstream evolution is slow. If L is a streamwise lengthscale, we are assuming $\delta/L << 1$, so that we can neglect variations in the streamwise direction compared to those in the vertical for averaged variables (i.e. $\partial/\partial x = 0$). Given these assumptions, the Reynolds averaged equations become,

$$\bar{w}\frac{d\bar{u}}{dz} = \frac{d}{dz}\left(\nu\frac{d\bar{u}}{dz} - \overline{w'u'}\right), \qquad \frac{d\bar{w}}{dz} = 0. \tag{20.3}$$

Because of the no normal flow through the boundary, we have $\bar{w} = w' = 0$ at $z = 0$, the bottom boundary. Then from eq. (20.3b) $\bar{w} = 0$ for all z. Then eq. (20.3a) becomes,

$$\frac{d}{dz}\left(\nu\frac{d\bar{u}}{dz} - \overline{w'u'}\right) = 0. \tag{20.4}$$

Hence if we have a stress τ given by,

$$\tau = \nu\frac{d\bar{u}}{dz} - \overline{w'u'} = \left(\nu\frac{d\bar{u}}{dz}\right)_{z=0}, \tag{20.5}$$

this stress is constant throughout the boundary layer. Near the boundary the stress is dominated by the viscous term. Away from the boundary we will have,

$$\tau = -\overline{w'u'}. \tag{20.6}$$

We can define a velocity scale from this surface stress

$$u_*^2 = \tau, \tag{20.7}$$

where u_* is the **friction velocity**. Away from the boundary eq. (20.6) implies that u_* is the turbulent velocity fluctuation magnitude.

20.1.2 Viscous sublayer: law of the wall

The frictional length scale z_f is the scale at which $Re = 1$, i.e. the scale at which the viscous and turbulent stresses are of comparable magnitude. Thus the frictional length scale can be defined as,

$$z_f = \frac{\nu}{u_*}. \tag{20.8}$$

This lengthscale determines the transition between the inertial and viscous sublayers.

In the viscous sublayer $z < z_f$, the velocity must depend on z, the distance from the wall, u_*, the friction velocity and ν, the viscosity. We can write this relationship as,

$$\frac{\bar{u}}{u_*} = f\left(\frac{zu_*}{\nu}\right) \tag{20.9}$$

Note that $\bar{u}$ has been nondimensionalized by u_*, and the distance z has been nondimensionalized by the frictional lengthscale ν/u_*. We can rewrite the relation in nondimensional form,

$$\bar{u}^+ = f(z^+) \tag{20.10}$$

where $\bar{u}^+ = \bar{u}/u_*$ and $z^+ = zu_*/\nu$.

Near a rough wall, the characteristic scale instead of being controlled by a frictional scale, it may be controlled by roughness length z_0, if $z_0 > z_f$, and the self-similar solution in eq. (20.10) must be interpreted with $z^+ = z/z_0$.

20.1.3 Turbulent boundary layer: velocity defect law

Outside the viscous sublayer, we can neglect viscosity. Thus the only dimensional parameters that enter in the problem are the turbulent velocity scale u_*, the total depth of the boundary layer δ, and the height z away from the wall. We can express this dependence as,

$$\frac{d\bar{u}}{dz} = \frac{u_*}{\delta} g\left(\frac{z}{\delta}\right). \tag{20.11}$$

This relationship states that the mean velocity gradient, $d\bar{u}/dz$, which is the reciprocal of a transverse time scale for the mean flow, has to be of order u_*/δ and varies on spatial scales of order δ. Notice that we cannot make a similar scaling argument for the mean velocity $\bar{u}$ and say that $\bar{u} = u_* g(z/\delta)$, because the mean velocity depends on an additional external parameter, the velocity outside the boundary layer $\bar{u}_0$. We know that for $z/\delta \to \infty$, we have $\bar{u} \to \bar{u}_0$.

We can now integrate from $z = \infty$ in toward the boundary to obtain $\bar{u}$,

$$\int_z^\infty \frac{d\bar{u}}{dz'} dz' = \frac{u_*}{\delta} \int_z^\infty g\left(\frac{z'}{\delta}\right) dz', \tag{20.12}$$

and hence,

$$\bar{u}(z) - \bar{u}_0 = u_* F\left(\frac{z}{\delta}\right), \tag{20.13}$$

or in nondimensional form,

$$\bar{u}^+ - \bar{u}_0^+ = F(\zeta), \tag{20.14}$$

where $\zeta = z/\delta$. This is a similarity solution for $\bar{u}^+$, which assumes that as the boundary layer changes size, or for different boundary layers $\bar{u}^+$ has the same form. This similarity solution is only valid outside of the viscous boundary layer, and cannot satisfy the boundary condition $\bar{u} = 0$ at the wall.

20.1.4 Inertial sublayer: logarithmic layer

Thus far we have two different laws for $\bar{u}^+$. One applies close to the wall in the viscous sublayer and satisfies the no-slip condition $\bar{u} = 0$. The other applies further away from the wall and is not guaranteed to satisfy the no-slip boundary condition at the wall; actually it turns out that away from the wall $u_* \ll \bar{u}_0$ and thus $\bar{u} - \bar{u}_0 \approx -\bar{u}_0$. This indicates that a viscous sublayer with very steep gradients is required in order to satisfy the boundary conditions. Of course the velocity doesn't suddenly jump from one scaling behavior to another - there is a transition region. In this transition region we expect both the law of the wall and the velocity defect law to apply.

From eq. (20.10) we expect that,

$$\frac{d\bar{u}^+}{dz^+} = \frac{df}{dz^+}. \tag{20.15}$$

From eq. (20.14) instead we have,

$$\frac{d\bar{u}^+}{dz^+} = \frac{d\zeta}{dz^+}\frac{dF}{d\zeta} = \frac{\zeta}{z^+}\frac{dF}{d\zeta}, \tag{20.16}$$

where we used the fact that $\zeta = z/\delta$ and $z^+ = zu_*/\nu$. In this overlap region these two expression must be equal so,

$$\frac{df}{dz^+} = \frac{\zeta}{z^+}\frac{dF}{d\zeta}, \tag{20.17}$$

and rearranging terms,

$$z^+\frac{df}{dz^+} = \zeta\frac{dF}{d\zeta}. \tag{20.18}$$

The right hand side of eq. (20.18) depends only on ζ and the left hand side can depends only on z^+. This can only be true only if both sides are equal to a constant,

$$z^+\frac{df}{dz^+} = \zeta\frac{dF}{d\zeta} = \frac{1}{\kappa}, \tag{20.19}$$

where κ is the **Von Karman constant**. This implies that

$$\frac{d\bar{u}}{dz} = \frac{u_*}{\kappa z} \tag{20.20}$$

so that in this region the only important quantities are u_* and z. Then in this transition region, the inertial sublayer, the flow is unaware both of viscosity and of the size of the boundary layer δ - just as in the inertial range isotropic homogeneous 3D turbulence is unaware of viscosity or of the integral scale of the forcing.

Integrating eq.(20.19) we have,

$$\frac{\bar{u}}{u_*} = \frac{1}{\kappa}\log\left(\frac{u_* z}{\nu}\right) + C_1, \tag{20.21}$$

and ,

$$\frac{\bar{u}-\bar{u}_0}{u_*} = \frac{1}{\kappa}\log\left(\frac{z}{\delta}\right) + C_2. \tag{20.22}$$

The region where this applies ($\zeta \ll 1$, $z^+ \gg 1$) is known as the logarithmic layer.

Near a rough boundary, the equivalent of 20.21 would be,

$$\frac{\bar{u}}{u_*} = \frac{1}{\kappa}\log\left(\frac{z}{z_0}\right) + C_1, \tag{20.23}$$

with z_0, the roughness length, taking the place of $z_f = \nu/u^*$, the frictional lengthscale.

Hinze (chapter 7, pag. 477) in figure 1.6.1 show the mean velocity distribution adiajent to a smooth wall, showing the logarithmic distribution away from the viscous region next to the wall and the linear region in the viscous sublayer from a composite of different laboratory experiments.

The value of the Von Karman constant has been measured in a variety of laboratory flows that indicate a universal value of 0.41. Some early measurements in the atmosphere (Businger et al., 1971) suggested a much smaller value of 0.35, and this led to speculations for a while that the constant might not be universal, but instead a function of salient nondimensional numbers in the flow (for example the Rossby number). Careful reexamination of the errors involved (Hogstrom, 1996) and more recent observations (Zhang, 1988) indicate that the constant is indeed a constant with a value around 0.40 ± 0.01.

20.2 Shear turbulence in stratified boundary layers

See Benoit Cushman-Roisin, section 11-1.

20.3 Planetary Boundary Layers

The boundary layers in geophysical flows are also affected by rotation through the Coriolis force. This is discussed by Tennekes and Lumley, chapter 5.3.

20.4 Convection

Convection is the process by which vertical motions modify the buoyancy distribution in a fluid. In the example considered above, the mixing of the upper ocean layer is caused by the mechanical action of the wind stress, and convection is said to be forced. Free convection arises when the only source of energy is of thermodynamic origin, such as an imposed heat flux. A common occurrence of free convection in geophysical fluids is the development of an unstable atmospheric boundary layer.

Glenn showed that free convection occurs when the Rayleigh number Ra,

$$Ra = \frac{\Delta b h^3}{\nu \kappa_T}, \tag{20.24}$$

exceeds a critical value, which depends on the nature of the boundary conditions. For a fluid confined between two rigid plates and maintained at different tempertures at the two plates, the critical Rayleigh number is $Ra = 1708$. At values slightly over the threshold, convection organizes itself in parallel two dimensional rolls or in packed hexagonal cells. At higher values of the Rayleigh number, erratic time dependent motions develop, and convection appears much less organized.

Geophysical flows almost always fall in this last category, because of the large depths involved and the small values of molecular viscosity and diffusivity of air and water. In the atmospheric and oceanic boundary layer, where the Rayleigh number easily exceeds 10^{15}, convection is manifestly turbulent and viscosity/diffusivity play secondary roles. In this limit, the usntably-stratified part of the water column mixes to become essentially uniform. For fixed buoyancy boundary conditions, thin layers develop near the boundaries with thickness such that the local Rayleigh number is nearly critical. if the flux of buoyancy is fixed, these layers do not occur and the buoyancy gradient decreases to small values.

20.5 Ocean Mixed Layer Models

20.5.1 Bulk Mixed Layer Models: Price Weller and Pinkel Model

Price, Weller, and Pinkel (PWP) proposed a simplified boundary layer model for the upper ocean. The model is based on simple heuristic arguments and has proved quite accurate. The model adjusts the distributions of momentum and tracer properties, and in doing that it sets the mixed layer depth. The internal workings are rather simple. After adding the surface forcing, one applies three criteria for vertical stability (i.e. whether water should mix vertically, and whether the mixed layer should deepen). After that, it applies advection and diffusion (vertical advection and vertical diffusion) to the water column.

Static Stability Criterion

The first stability criterion, and the one that proves the most important in the model, is static stability. In fact, it accounts for about 80% of the "action". Quite simply put, one cannot have denser water overlying lighter water. This means that one must have $\partial_z \bar{b} \geq 0$. Thus one goes through the model domain (let "i" be the position index, with "i" increasing downward), one tests to make sure that,

$$\bar{b}_i \leq \bar{b}_{i+1}, \tag{20.25}$$

and where this is not the case, one then mixes all the cells above this depth (that is average them among themselves). In general, what one should really do is to just mix the two cells together, then start from the top of the model and do it again. What happens in practice, however, is since all the heat exchange (in particular cooling, which decreases buoyancy) takes place at the top of the model, one always finds that the effect of this instability is to mix all the way back to the top. So one may as well do it the first time. This scheme is equivalent to the convective overturning scheme described above, if one sets the diffusivity to infinity whenever there is static instability.

Bulk Richardson Number Stability Criterion

The second stability criterion is the bulk Richardson Number stability. This arises due to the fact that if the mixed layer gets going too fast (i.e. the wind stress is allowed to accelerate it to too great a speed), it tends to "stumble" over itself. What actually happens is that if there is too much velocity shear at the base of the mixed layer, it will tend to mix downward. This effect, determined by field and laboratory experiments is such that the mixed layer deepens if the bulk Richardson number goes below a critical value,

$$R_b = \frac{h\ \Delta\bar{b}}{|\Delta\bar{\boldsymbol{u}}|^2} \geq 0.65, \tag{20.26}$$

where h is the height (thickness) of the mixed layer, $\Delta\bar{b}$ is the buoyancy contrast between the mixed layer and the water below, and $\Delta\bar{\boldsymbol{u}}$ is the difference in horizontal velocity between the mixed layer and the underlying water. This effect tends to be important when the mixed layer becomes very thin, because a thin mixed layer becomes easily accelerated by wind stress, and the inverse quadratic nature of the dependence makes for a strong damping. The relative activity of this process is about 20% of the static instability.

Gradient Richardson Number Stability Criterion

The third stability criterion is based on the gradient Richardson number, and has the effect of stirring together layers where the velocity gradient becomes too great.

One can think of this as the mixed layer "rubbing" against the water underneath it. This largely has the effect of blurring the transition between the mixed layer and the seasonal thermocline below, which would normally be rather sharp. Laboratory experiments indicate that there is a critical gradient Richardson number, below which stirring occurs,

$$R_g = \frac{\partial_z \bar{b}}{|\partial_z \bar{\boldsymbol{u}}|^2} \geq 0.25. \tag{20.27}$$

This turns out to be a not very vigorous process, but becomes a little more important in the absence of any explicit turbulent vertical diffusion. Notice that the gradient Richardson number introduced by Price, Weller, and Pinkel differs from the one used in KPP in that it does not include any parameterization for unresolved turbulent shear.