

Wind-Driven Circulation: Stommel's gyre & Sverdrup's balance

We begin by returning to our system of equations for flow of a layer of uniform density on a rotating earth.

$$\frac{du}{dt} - fv = -g \frac{\partial h}{\partial x} - ru + \frac{F^x}{\rho},$$

$$\frac{dv}{dt} + fu = -g \frac{\partial h}{\partial y} - rv + \frac{F^y}{\rho}$$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}[u(H+h)] + \frac{\partial}{\partial y}[v(H+h)] = 0, \text{ where}$$

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y}$$

We recall that $z = -H$ is the bottom of the fluid layer and $z = h$ is the free surface. For simplicity, we will define D to be the total depth of the fluid: $D = H+h$. Because the bottom of the fluid is not changing in time, we can replace the time rate of change of h in the third equation with the time rate of change of D . Next we define vorticity.

An important fluid property associated with shearing and rotating elements of motion is vorticity, ζ . It is defined as follows:

$$\zeta \equiv \nabla \times \mathbf{v}$$

It is a vector, but we will be considering here the vertical component of vorticity, which is

$$\zeta^z = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$$

This quantity depends on the shears in the flow, not the flow itself. We will see that is an important quantity in a moment. But first note in the above equations that momentum can change following a fluid particle because of external forces, friction and pressure gradients. The latter element of non-conservation can be removed by considering vorticity because if one differentiates the second equation above by x , the second equation above by

y and then subtracts them, the term associated with pressure gradients can be eliminated. We will not derive the following equation here, but you can do this on your own [in fact it may be a new homework exercise], but by using the above procedure and the definition for vorticity, and all three of the above equations, we can obtain the following without *any* assumptions about the Coriolis parameter being constant (except in time!):

$$\frac{d}{dt} \left[\frac{f + \zeta^z}{D} \right] = -\frac{r}{D} \zeta + \frac{1}{\rho D} (F_x^y - F_y^x) = -\frac{r}{D} \zeta + \frac{1}{\rho D} (\nabla_h \times \vec{F})$$

In the absence of friction or external forces, the quantity $[(f + \zeta)/D]$ is conserved, where we have now dropped the subscript “v” for vertical, which is assumed. This quantity is called ***potential vorticity*** and is one of the fundamental properties used in physical oceanography. It can be changed by friction and by a curl of an external force (e.g. wind stress) but otherwise is remains constant. It is so important, that it is even known by the cartoon character “Dilbert” (see figure shown in class). It states that, in the absence of external forces (or friction), fluid will tend to follow contours of constant depth in the ocean. Since depth variations occur much more rapidly than variations of the Coriolis parameter, fluid parcels can cross depth contours only by developing a substantial relative vorticity.

Concept of linearization:

If the equations are examined, not all terms are always of the same size. We have already recognized that overall, friction will be small and for many of the flows the basic balance will be geostrophic, for example. Because the equations are non-linear, this makes solution very difficult. SO if some of the small terms are the non-linear ones, ignoring them can allow us to obtain a solution! Consider the quantity potential vorticity, for example. It contains quantities which have different magnitudes. We will now look at some of these terms [this exercise is called “scaling”].

We can denote the scale of a quantity by square brackets:

$$f \approx [f]$$

$$\zeta \approx [u/L]$$

$$h \approx [h]$$

$$H \approx [H_0]$$

$$\left[\frac{f + \zeta}{H + h} \right] \approx \left[\frac{f}{H_0} \right] \left[\frac{1 + (u/fL)}{1 + (h/H_0)} \right] \approx \left[\frac{f}{H_0} \right], \text{ if}$$

$$R \equiv [u/fL] \ll 1, \text{ and } [h/H_0] \ll 1$$

The quantity R , is called the Rossby Number and is a ratio of the relative vorticity, ζ to the planetary vorticity, f . For most velocities and for most horizontal length scales we will use in the course, this is a pretty good approximation (e.g. $R \ll 1$). If the total depth of the fluid is much larger than the free surface elevation changes, then the second approximation is good as well. In this case, potential vorticity simplifies to (f/H) which, in the absence of other forces, must be conserved. What this means is that the fluid must always flow along lines of constant (f/H) . For an f -plane, this means that fluid cannot cross depth contours without some external force operating. This is a pretty powerful statement!

Now we will look at Stommel's (1948) examination of wind-driven gyres. We have already derived everything we need. We will use the above linearizations to simplify the potential vorticity (PV) equation. His model was for an ocean of constant depth, so this simplifies things even more. Since H is constant and f varies with latitude (y -direction), this means that PV conservation requires that fluid can flow only east/west NOT north/south unless there are external forces. This is now

$$\frac{d}{dt} \left[\frac{f + \zeta}{D} \right] = -\frac{r}{D} \zeta + \frac{1}{\rho D} (F_x^y - F_y^x), \text{ or after scaling}$$

$$\beta v = -r(v_x - u_y) - \frac{1}{\rho} (F_x^y - F_y^x)$$

With constant depth, $(u_x + v_y = 0)$ and we can write the second equation above in terms of a stream function ψ : $(u, v) = (-\psi_y, \psi_x)$

$$r(\psi_{xx} + \psi_{yy}) + \beta\psi_x = -\frac{1}{\rho}(F_x^y - F_y^x)$$

Stommel used a simplified form of the external forcing as well: it was a sinusoidally-varying body force in the “x-direction, but varying only in the “y direction”. At this stage, one can associate this roughly with a meridionally-varying zonal wind stress. If this force is given by $(F^x, F^y) = ((-a\pi\rho/b)\cos(\pi y/b), 0)$, using Stommel’s notation, then the above becomes

$$r(\psi_{xx} + \psi_{yy}) + \beta\psi_x = a \sin(\pi y/b)$$

This is the equation Stommel solved which showed the different types of flow depending on the rotation (or non-rotation) of the earth, and due to the earth’s curvature. The solutions are shown in the textbook on p. 94. We will discuss this in class. Because β is always positive, the intense western boundary flow is *always* on the western boundary. If the wind stress is reversed, the sense of rotation of the flow will be reversed but the strong boundary flow will be to the south along the western boundary. This is what we would expect in the subpolar gyres (to the north of the subtropical gyre Stommel considered. This general result explains a lot about the structure of the wind-driven circulation in both hemispheres. You might think through what happens in the southern hemisphere where f is negative (but NOT β).

Sverdrup’s balance

If we examine Stommel’s solution for the flow in the case of rotation on a β -plane on p. 94, lower left, we see that except for the region near the western boundary of the basin, where the flow is strong and to the north, the flow everywhere else is zero or to the south. We can think of this interior region as somehow having different dynamics than the western boundary current (wbc) region. If one looks at the previous equation for the streamfunction of the flow, we see three terms: one proportional to friction, one having β , and a third with the forcing. The first term becomes important near the western boundary where gradients in the streamfunction are large. Elsewhere friction is less important because gradients of the streamfunction (actually second derivatives!) are small. So one might imagine an *interior* region in which the second two terms are important and a *boundary layer* region in which the first two terms are important, the forcing there having no strong variation.

The interior region is one that Sverdrup studied in 1947 and the balance of these two terms has become known by his name: The Sverdrup Balance. If we re-write this balance slightly, it's value will become more apparent.

$$\beta \psi_x = \frac{1}{\rho} (\nabla_h \times \vec{F}), \text{ or}$$

$$\beta \iint dx dz v = \frac{H_0}{\rho} \int_x dx (\nabla_h \times \vec{F})$$

If we integrate the first equation across a basin from east to west at any fixed latitude ($y=\text{constant}$) and then multiply by the depth of the fluid, we obtain an estimate (second equation) for the interior wind-driven transport across the basin in the meridional direction. The direction of this transport depends on the sign of the curl of the forcing, F . In the case of the Stommel forcing, the curl of the forcing is negative and thus the interior flow is to the south. So we can then estimate, using knowledge about the forcing, what the wind-driven circulation must look like. Whatever interior flow is demanded by the Sverdrup balance must then be returned in the wbc as a boundary layer flow. With knowledge about the wind stress and its curl, one can make direct estimates of the wind-driven flow using the Sverdrup Balance. Below we have extracted two figures from a manuscript written by Josey, Kent, and Taylor (JPO, submitted) that discusses wind stress from a couple of well-known climatologies (Hellermann and Rosenstein, JPO, 1983; and Josey, Kent & Taylor, 1998, from Southampton Oceanography Centre [SOC]).

Image removed due to copyright concerns.

In the first figure (above) , we show some global maps of wind curl from these two climatologies (based on ship observations over many years). Regions of negative (positive) curl are subtropical gyres in the northern (southern) hemisphere. We are not interested in the differences in these two climatologies at this point: you can read all about this in the paper on the web at <http://www.soc.soton.ac.uk/JRD/MET/PDF/SOCHR.pdf> . What we are interested in here are the patterns and in the zonal integration of the curl field to produce a volume transport streamfunction following the Sverdrup Balance, which is shown below (also from the same reference).

Image removed due to copyright concerns.

In case you are wondering about the units in the second figure, they are in Sverdrups ($1\text{ Sv} = 10^6 \text{ m}^3/\text{s}$, also named after guess who?). As a point of comparison, the Amazon river, which transports the most water of any river in the world, has a mean volume transport of 0.3 Sv. So the wind-driven flow drives transports in the ocean that are a hundred times larger than the Amazon outflow. As we will see later, the actual transports in wbc's are substantially larger than what is anticipated by the Sverdrup balance. But this is not going to deter us & we will not throw out this valuable tool for understanding the wind-driven circulation.

The Sverdrup circulation is constructed by beginning the integration of the wind curl at the eastern boundary of all oceans and requiring that there be no flow into the ocean through/from the *eastern* boundary. You will see that there is a lot of flow into and out of the boundary on the *western* sides of the ocean basins. This is because we need western boundary currents to return the wind-driven flow in these regions and they are not part of the dynamical balance. Finally, note that while assumptions of geostrophic motion break down at the equator, there is nothing in the Sverdrup Balance that breaks down there. So unlike circulation maps inferred from dynamic height, maps of the Sverdrup circulation can be used to infer wind-driven circulation on and across the equator.

Compare the subtropical gyres in the N. Atlantic and N. Pacific. The winds are similar in the two basins and so are the wind stress curls. But because the N. Pacific is nearly twice as wide zonally as the Atlantic, the Sverdrup transport (which is the zonal integral of the wind curl) is nearly twice as large. The two wind climatologies used above are quite different in the Southern Ocean. Problems there are largely because there are so few ship observations in that part of the ocean. Another problem is because there are no continental boundaries over the latitude zone defined by Drake Passage (between S. America and the Antarctic peninsula). Therefore we can't define a zonally-integrated Sverdrup Balance there as there are no eastern or western boundaries to start from and end at!

Before we return to the N. Atlantic subtropical gyre, one further digression is needed. We have been rather vague about how the body force, F , is actually related to the wind stress. We have also been rather vague about the nature of the frictional force. As these are two of the three terms in the wind-driven theory of Stommel, it is probably appropriate to spend some time on this now. They are both related to the way that stress is transferred between fluid particles in the ocean. Once we have done this, we can also briefly discuss Munk's (1950) paper on the wind-driven circulation and how it differs from Stommel's.