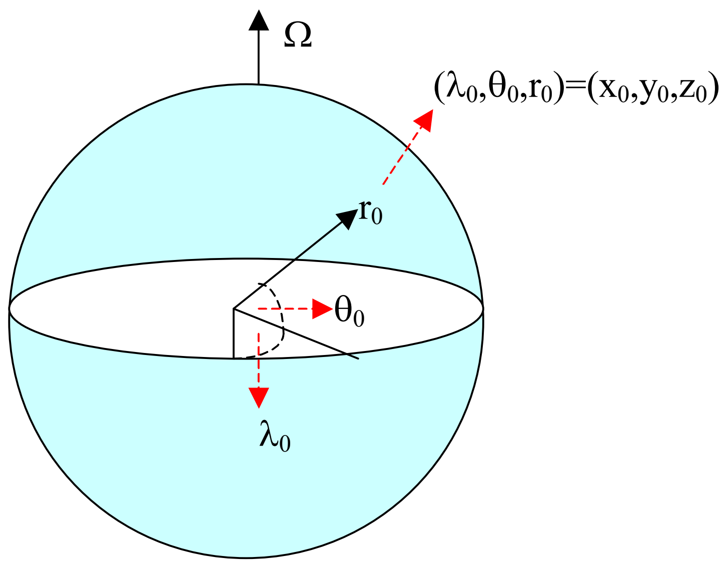
Rotation 101 (cont.): Effects of rotation on a sphere: f- and β-planes

We will follow the convention of many before us and try to simplify the effects of the curvature of the earth on the dynamics of particles in this rotating system. An approximation is made in which the position of a particle in a **spherical geometry** (λ,θ,r) is expressed in terms of **cartesian** variables (x,y,z) for small excursion from a reference point.

Consider the following figure.

On the surface of the sphere, we choose a reference point at a longitude, latitude, and radius denoted by the coordinate (λ_0,θ_0,r_0), with the Cartesian coordinates of (x_0,y_0,z_0). For *small* distances from this point, we can write the following:



$$x' = x - x_0 = r_0 \cos\theta_0(\lambda - \lambda_0)$$
$$y' = y - y_0 = r_0(\theta - \theta_0)$$
$$z' = z - z_0 = r - r_0$$

Gravity is in the radial or "z" direction and the rotation vector **Ω**, velocity **v**, and Coriolis acceleration 2 **Ω x v** have the following components:

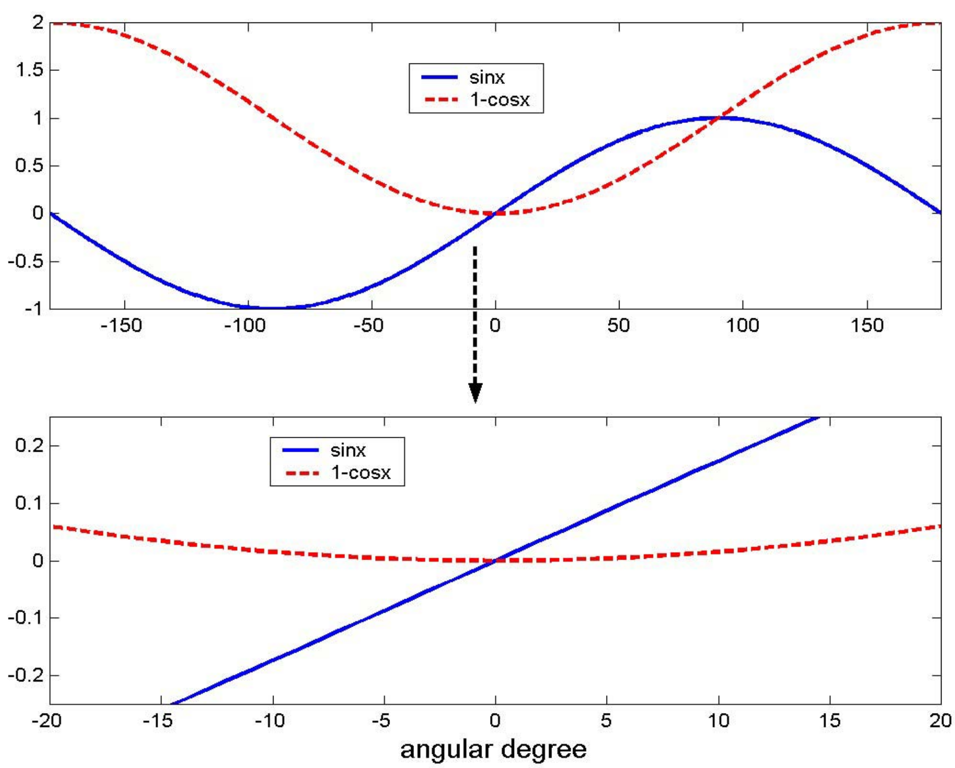
$$\vec{\Omega} = (0, \Omega\cos\theta, \Omega\sin\theta),$$
$$\vec{v} = (u, v, w) = (\frac{dx'}{dt}, \frac{dy'}{dt}, \frac{dz'}{dt})$$
$$2\vec{\Omega}\times\vec{v} = (2w\Omega\cos\theta - 2v\Omega\sin\theta, 2u\Omega\sin\theta, -2u\Omega\cos\theta)$$

Because vertical velocities are much smaller than horizontal velocities for large-scale, low frequency motion, and because the basic balance in the vertical is hydrostatic, the last equation above can be approximated as

$$2\vec{\Omega}\times\vec{v} = (-2\Omega v\sin\theta, 2\Omega u\sin\theta, 0) = (-fv, fu, 0), \textit{where}$$
$$f \equiv 2\Omega\sin\theta$$

the Coriolis parameter, f, is the vertical component of the rotation vector. This approximation will shaky in high latitude regions where vertical convection in winter can create large vertical velocities and at the equator where large zonal currents can affect the hydrostatic balance.

One problem with the Coriolis parameter above in trying to understand the simplified physics of the ocean is that it is not a constant [$sin\theta$ varies with latitude]. In the problems we will consider in this course, the ocean basin can be considered small enough so that locally, some further simplifications can be made. We will consider two examples called the "**f-plane**" and "**β–plane**" approximations.



We now insert into the above the fact that $\theta = \theta_0 + \theta'$, where $\theta' \ll 1$, & use the trigonometric identity for *sin(a+b)=sin(a)cos(b)+cos(a)sin(b)*. Now as seen in the above figure, for small angular changes (expressed in radians) from the reference latitude, $\cos(\theta') \sim 1$ & $\sin(\theta') \sim \theta'$.

Thus for the vertical component of the rotation vector, f, we get the following:

$$f = 2\Omega \sin(\theta_0 + \theta') \approx 2\Omega \sin\theta_0 + \frac{2\Omega\cos\theta_0}{r_0} r_0(\theta - \theta_0)$$

$$= f_0 + \beta_0\, y', \ \textit{where}$$

$$f_0 \equiv 2\Omega \sin\theta_0, \ \textit{and}$$

$$\beta_0 \equiv (2\Omega\cos\theta_0 / r_0)$$

An "**f-plane**" approximation assumes that the vertical component of the rotation vector is constant. The **"β-plane"** approximation assumes that f is constant unless it's y-derivitave is taken, in which *df/dy* is replaced by another constant equal to β.

The homework problem "pucks_on_ice" is done on an f-plane; our work on the wind-driven circulation will require use of the β-plane. The following material will be needed when we begin the section on the ocean circulation, chapter 3 in the text.

Fluid equations for barotropic and baroclinic flows

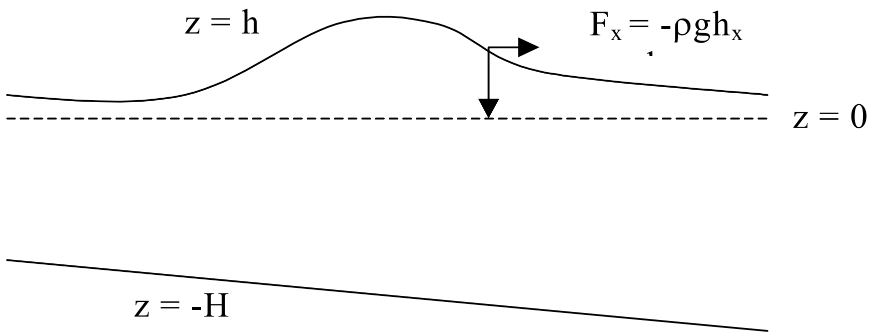
Thus far we have dealt with solid objects moving around on a solid surface with little or no fluidity. But in fact we have almost derived all of the equations needed for the study of a layer of fluid of uniform density on the surface of the spherical earth! There is one last item needed to make the transition from pucks on ice to the general circulation of the ocean: it is that bumps on the surface of a fluid will not stay there: motion will be away from the bump and its surface expression will vanish unless sustained by some force.

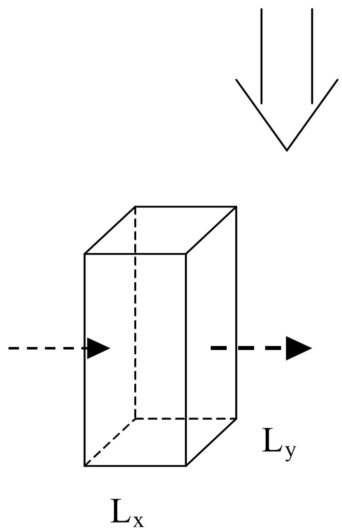
For those of you with no experience in fluid dynamics, this may appear to be a formidable hurdle to overcome. We have tried to make this as painless as possible: starting with pucks on a frozen ocean, yet ending up with the equations physical oceanographers use to understand their observations. For those with fluids in their past, this may seem less rigorous than the traditional method of derivation of the Navier-Stokes equations for a rotating fluid. In this course, we will not be trying to solve the complete sets of fluid motion in either the barotropic or baroclinic limits. In fact, this is impossible as non-linearity and time dependence due to turbulence precludes *any* general solution. So don't worry about solving them! We will simplify them for particular problems, however, and want them exposed now for later reference.

Consider the following diagram.

A layer of fluid of density ρ lies over a sloping bottom (z=-H) and a free surface. Because of a bump of fluid on the surface, there will be pressure gradients created that try to force the underlying fluid away from the surface bump. In the lower panel, we consider a column of fluid of sides L_x,L_y and height H+h (the total depth of the fluid).

Fluid can escape the box by flowing thru the imaginary sides, but not through the bottom, which is solid. Because fluid can leave the box, the free surface level can change. Here we do a mass (or volume balance) for the box, whose volume, V is $L_x L_y(H+h)$.





$uL_y(H+h)_{rhs} - uL_y(H+h)_{lhs} = L_y\Delta[u(H+h)]$ = net loss of fluid in x-dir.
$vL_x(H+h)_{front} - vL_x(H+h)_{back} = L_x\Delta[v(H+h)]$ = net loss of fluid in y-dir.,

where *lhs* refers to the "left hand side" of the box. This net loss of fluid must be balanced by a change in volume. So we can write a volume balance as follows (first simplifying by letting L_x=L_y=L):

$$\frac{\Delta V}{\Delta t} + L[\Delta(u(H+h)) + \Delta(v(H+h))] = 0$$

$$\frac{\Delta V}{\Delta t} = L^2\frac{\Delta h}{\Delta t}, thus$$

$$\frac{\Delta h}{\Delta t} + \frac{\Delta(u(H+h))}{L} + \frac{\Delta(v(H+h))}{L} = 0, or\, as\, L \to 0$$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}[u(H+h)] + \frac{\partial}{\partial y}[v(H+h)] = 0$$

So we now have the following equations of motion for a fluid ocean of uniform density:

$$\frac{du}{dt} - fv = -g\frac{\partial h}{\partial x} - ru + \frac{F^x}{\rho},$$

$$\frac{dv}{dt} + fu = -g\frac{\partial h}{\partial y} - rv + \frac{F^y}{\rho}$$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}[u(H+h)] + \frac{\partial}{\partial y}[v(H+h)] = 0, where$$

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$$

For fluids, we can either track particle motion [Lagrangian formulation] or look at force balances at a position, while different particles flow by [Eulerian formulation]. Thus far, we have done the former, but not using any fluid dynamics. When one does, the time derivatives take on different meaning in the two formulations. This is given in the last of the above equations, where a time derivative following a particle becomes an advective derivative at a point (x,y) in the Eulerian formulation. This reflects the fact that at a point, there can be changes in time because of external forces and because different particles with a different history are streaming by the point of observation. As far as any mathematics needed for the rest of this course, the difference is immaterial since we will be looking at simple linearized versions of the above equations. Yet we will be using them to understand the dynamics of mid-latitude gyres, the deep tropics and equator, and time dependent motions such as tides, Rossby, and topographic waves, in which we consider the whole water column responding as one with no density variations in the vertical. Motions of this nature have been called "**barotropic**" and are an important part of the ocean circulation.

However, we know that density does vary in the vertical and that ocean currents are not uniform with depth. The part of the velocity field that varies in the vertical is known and the "**baroclinic**" velocity, and our equations need to be slightly modified to account for this component. We will now briefly discuss what needs to be modified in our set of "grand equations" for the baroclinic flows.

First of all, we must recognize that *horizontal* gradients of pressure will act to cause *horizontal* acceleration of the fluid, subject to the presence of the Coriolis force. Recall that we have employed a hydrostatic balance in the vertical in which vertical gradients of pressure are balanced by gravity. So the additional equation is the hydrostatic balance, which states that

$$\frac{dw}{dt} = -g\rho - \frac{\partial p}{\partial z} = 0$$

In the event that the hydrostatic balance is upset, there will be vertical accelerations of the fluid.

The next modification to the equations of motion is that we need to replace horizontal variations of the ocean surface, with horizontal variations of pressure, as these can now vary in the vertical away from the ocean surface. So we have

$$g\frac{\partial h}{\partial x} \rightarrow \frac{1}{\rho}\frac{\partial p}{\partial x}, \quad g\frac{\partial h}{\partial y} \rightarrow \frac{1}{\rho}\frac{\partial p}{\partial y}$$

Next, we must recognize that the advective derivative must include vertical velocity and a "z" derivative. Thus

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

Finally, the last equation of the earlier group, also known as the continuity equation, expresses the fact that flows into a control box must be balanced by flows out of the volume or changes in the free surface. Away from the free surface, this must now be modified by considering a small cube of side L within the fluid. The volume flux across the six faces of the cube now becomes

$$\frac{\Delta V}{\Delta t} + L^2[\Delta(u) + \Delta(v) + \Delta(w)] = 0, where\ V = L^3, or$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, as\ L \to 0$$

since by definition, we have a fixed volume of fluid in the cube and *L* is constant on all sides of the cube. So for future reference, the grand set of equations which can account for vertical variations in the flow (and for the case in which there is no vertical flow variation) now become

$$\frac{du}{dt} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} - ru + \frac{F^x}{\rho},$$

$$\frac{dv}{dt} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} - rv + \frac{F^y}{\rho}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g,$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

We have now four equations for the five dependent variables (u,v,w,p,ρ) subject to external forces which have not yet been further defined at this stage. The frictional force, r, is subtle in that it is expected to be large near solid boundaries of the fluid and near the ocean surface where turbulence levels are large and can act to oppose or create motion. The usual method for understanding of friction would take us too far into fluid dynamics. For the present, understand that friction is important where lateral stresses applied at boundaries of the fluid can force the motion of the fluid, such as the top [where we have wind stress forcing the ocean], and on the bottom and sides of the ocean, where lateral stresses act to impede the motion of the fluid. The 5th equation (missing above) prescribes how density is affected by advection & external forces. We will be dealing with this later. For now, consider it prescribed!