

Internal Waves

We now examine internal waves, for the case in which there are two distinct layers and in which the lower layer is at rest. This is an approximation of the case in which the upper layer is much thinner than the deeper layer, but this makes the dynamics much simpler. The linearized equations of motion (for small perturbations) are as before, but now for the upper layer only:

$$\begin{aligned}\frac{\partial u_1}{\partial t} - fv_1 &= -g \frac{\partial h}{\partial x}, \\ \frac{\partial v_1}{\partial t} + fu_1 &= -g \frac{\partial h}{\partial y}, \\ \frac{\partial H_1}{\partial t} + H_1 \left[\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} \right] &= 0\end{aligned}$$

where H_1 is the total thickness of the upper layer. The upper surface, h , is free to move vertically, as is the bottom of the upper layer. Recall before that in this case, there can be no pressure gradients in the lower layer (layer 2) if it is to remain at rest. If perturbations to the internal layer interface are small, and its depth is otherwise at a constant H_{01} , then the vanishing of pressure gradients at any depth in the lower layer, $z = -z_0$, requires the following:

$$\begin{aligned}P_z(x, y, z_0) &= \rho_1 g (H_{01} + h_1 + h) + \rho_2 g (z_0 - (H_{01} + h_1)), \text{ and} \\ \nabla_{x,y} P_2 = 0 &\rightarrow h \approx h_1 (\rho_2 - \rho_1) / \rho_1 = h_1 \Delta \rho / \rho\end{aligned}$$

Upward displacements of the free surface must be accompanied by *downward* displacements of the interface to prevent pressure gradients in the lower layer. We have defined the displacements of the interface so that a *downward* displacement of the interface is equivalent to a *positive* value of h_1 . The reason for this will be apparent shortly. Since the density difference is going to be a small compared to the density of the surface layer ($\Delta \rho / \rho \ll 1$), changes in the free surface are going to be small compared to those of the interface. Thus, in the third equation above, we can ignore variations in the surface height compared to changes in the interface height. When we now substitute this approximation for h and H_1 into the above equations we obtain:

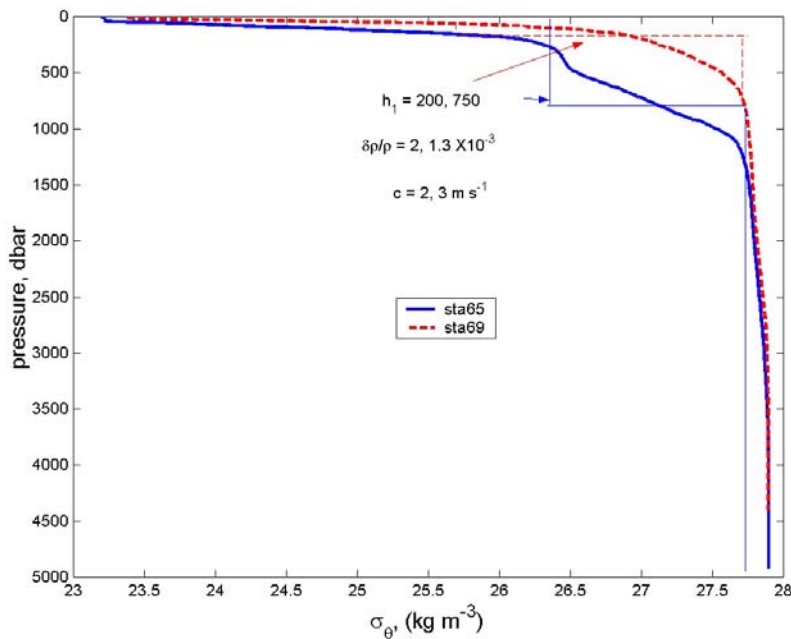
$$\frac{\partial u_1}{\partial t} - fv_1 = -g' \frac{\partial h_1}{\partial x},$$

$$\frac{\partial v_1}{\partial t} + fu_1 = -g' \frac{\partial h_1}{\partial y},$$

$$\frac{\partial h_1}{\partial t} + H_{01} \left[\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} \right] = 0, \text{ where}$$

$$g' \equiv g \frac{\Delta \rho}{\rho}.$$

Note that these are *exactly* the equations for a homogeneous fluid of uniform density with two exceptions: the water depth now becomes the mean depth of the upper layer ($H \rightarrow H_{01}$) and the gravity term becomes a reduced gravity ($g \rightarrow g'$). Thus all of the previous types of surface waves can be immediately taken over for internal waves with these two differences in mind. The small differences make some substantial changes in the properties of these waves. Consider two quantities which were important for all of the surface waves: the shallow water phase speed c and the Rossby Radius of deformation, a . These will take on different numerical values. We need to first estimate some basic quantities, such as the density ratio and the mean interface depth. For this, we will use two stations from either side of the Gulf Stream (they were used in one of the homework problems).



In the above figure, two hypothetical cases for dividing up the water column into two layers have been made. In the first, the station (#65) is south of the Gulf Stream in a thick layer of subtropical mode water (Eighteen Degree Water). We have used a surface density appropriate for what might be found in winter and a pycnocline depth of 750m. The stations were occupied in September when the seasonal pycnocline is fully developed. In the second, we used a shallower pycnocline (200m) but a larger density ratio (2×10^{-3}). The phase speed and radius of deformation are given by:

$$c^2 = g' H_{01},$$

$$a^2 = c^2 / f^2 = g' H_{01} / f^2$$

We see that the phase speed has been reduced from 200 m/s to 2-3 m/s and the radius of deformation is reduced from 2000km to 20-30km (for a value of f of 10^{-4} s^{-1} .) While this change in two of the key parameters governing wave motion is two orders of magnitude, there is no qualitative difference in the types of waves or their dispersion relations. We will briefly review this now.

Internal gravity/inertial waves:

$$\sigma^2 = f^2 + c^2 \kappa^2, \text{ where}$$

$$\kappa \equiv (k, l), \text{ \& } \kappa^2 = (k^2 + l^2), \text{ and}$$

$$c^2 = g' H_1$$

Kelvin waves:

$$(h, v) = (h(x), v(x)) e^{i(l y - \sigma t)},$$

$$c^2 \frac{\partial^2 h}{\partial y^2} - \frac{\partial^2 h}{\partial t^2} = 0, \text{ giving}$$

$$\sigma^2 = c^2 l^2, \text{ where}$$

$$c^2 = g' H_1$$

Rossby waves:

$$u, v, h \approx (\hat{u}, \hat{v}, \hat{h})e^{i(kx+ly-\sigma t)} = (\hat{u}, \hat{v}, \hat{h})[\cos(kx + ly - \sigma t) + i \sin(kx + ly - \sigma t)]$$

$$\sigma = -\frac{\beta a^2 k}{1 + \kappa^2 a^2}, \text{ where } \kappa^2 = k^2 + l^2, \text{ where}$$

$$a^2 = g' H_1 / f^2$$

A smaller radius of deformation, a , means, among other things, that baroclinic scales are much smaller than barotropic scales: the adjustment problem in which a column of upper layer fluid is released and allowed to adjust to the rotation of the earth will yield a much narrower geostrophic current and a smaller ‘scale’ for the slumping of the interface in the upper layer. Whether Poincaré (or internal/inertial gravity) waves or Rossby waves are ‘long’ or ‘short’ compared to the radius of deformation will be greatly affected: most of these that were ‘short’ for barotropic motion will be ‘long’ for baroclinic waves. The frequency gap between the lowest frequency internal wave and the highest frequency internal Rossby wave will be much larger, because the lower limit for the former is the same as before (still the inertial frequency) but the upper limit for the latter is much smaller: internal Rossby waves have much longer periods than their barotropic counterparts.

For example, we estimated earlier that the minimum period (maximum frequency) of a barotropic Rossby wave would be 3.6 days. For the baroclinic case just considered, this would become 360 days. In other words, we would expect no propagating baroclinic Rossby waves to have a period shorter than about a year!

Baroclinic motions can have large amplitude displacements of the pycnocline: of 100m. These can be seen at the ocean surface, although reduced in amplitude by a factor of $\Delta\rho/\rho$. The satellite altimeter has provided the best view of these baroclinic waves because though the amplitudes be small (10 cm), the periods are long and thus easily sampled by the 10 day repeat orbit of the Topex-Poseidon altimeter. In a paper by Chelton and Schlax (Science, 1996). They showed a space-time series of altimeter variability at selected latitudes in the North Pacific Ocean, which we will now discuss.

In the diagram at the right, altimeter data have been filtered to look at signals having periods between 0.5 and 2 yrs. For the three years of data displayed, surface highs (depressions of the pycnocline) and lows can be seen to be

propagating to the west, most clearly in the western part of the Pacific. These have phase speed of order 10 cm/s and are consistent with speeds of the lowest mode baroclinic Rossby wave (similar to our approximation above). While the phase speeds seem slightly faster than expected, and this has been a focus of much debate in the literature, the evidence for the existence of these waves and their importance to long-term variability in the ocean is undisputed. Also evident is the evidence that baroclinic waves propagate faster towards the equator. This can be understood in terms of the variation of the radius of deformation with latitude: in the tropics, where f is smaller, the radius of deformation is larger than in mid-latitudes. Since long Rossby waves propagate have a phase speed that is proportional to the square of the radius of deformation ($c_p = \beta a^2$), as one approaches the equator, waves will take a shorter time to cross a basin than at higher latitudes. At 21N above, it will take about 4 years for a wave to cross the domain plotted, whereas it will take approximately 10 years at 39N. Of course the time series is not that long and we have to extrapolate to see this.

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Internal waves are responsible for much of the high frequency variability in open ocean observations. This can be deduced by the relationship between observed horizontal and vertical velocities and how they depend upon frequency. In shallow water, such as Massachusetts Bay, internal waves in the seasonal thermocline (pycnocline) are generated by tidal flow over Stellwagen Bank and can produce a visible signature at the ocean surface (due to surface convergences and divergences and their effect on surface waves) and substantial mixing at the ocean bottom. In fact, they can be a major source of nutrient supply to the upper ocean as these steep waves mix and occasionally break. A good description of these can be seen in a paper by Haury, Briscoe and Orr (Nature, 1979), which will be discussed in class.