## 5. Potential vorticity

Consider two closed material curves on two adjacent surfaces along which some state variable (e.g. $\sigma$ or $\theta_v$) is a constant, as illustrated in Figure 5.1. We will suppose that the area enclosed by the curves is infinitesimal, as is the distance between the two $s$ surfaces, and that the curves are connected by material walls so as to form a material volume, which is material only in the sense that points on its surface move with the component of the total fluid velocity projected onto $s$ surfaces, as with the derivation of the Circulation Theorem. Thus, material may leave or enter the *ends* of the material volume, but not the sides.

The amount of mass contained in the volume is

$$\delta M = \rho \, \delta A \, \delta n, \tag{5.1}$$

where $\rho$ is the fluid density and $\delta n$ is the distance between adjacent $s$ surfaces.

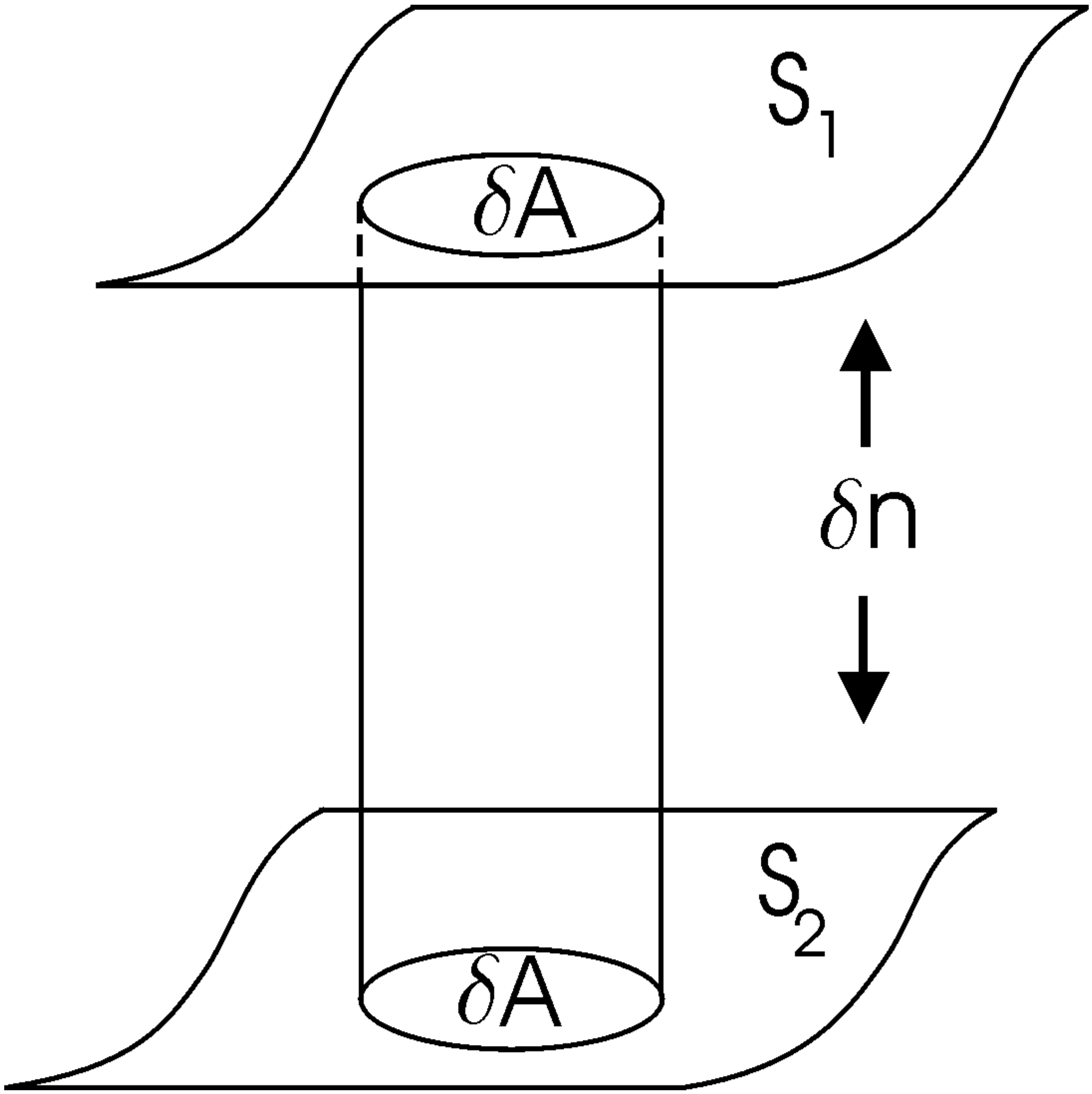


**Figure 5.1**

Since $n$ lies in the direction of $\nabla s$,

$$\delta n = \frac{dn}{ds}\delta s, \tag{5.2}$$

where $\delta s = s_2 - s_1$ is the difference between the values of $s$ on the two surfaces.

Using (5.1) and (5.2), the incremental area, $\delta A$, may be written

$$\delta A = \frac{1}{\rho \delta s} \frac{ds}{dn} \delta M. \tag{5.3}$$

Using this, the Circulation Theorem in the form (4.8), with the integrals taken over the infinitesimal areas, can be written

$$\frac{d}{dt} \left[ (\nabla \times \mathbf{V} + 2\mathbf{\Omega}) \cdot \hat{n} \frac{1}{\rho \delta s} \frac{ds}{dn} \delta M \right] = (\nabla \times \mathbf{F}) \cdot \hat{n} \frac{1}{\rho \delta s} \frac{ds}{dn} \delta M. \tag{5.4}$$

Given that the direction of $\hat{n}$ is the same as that of $\nabla s$, and that $\delta s$ is by definition a fixed increment in $s$, (5.4) may be re-expressed

$$\frac{d}{dt} \left[ \alpha (\nabla \times \mathbf{V} + 2\mathbf{\Omega}) \cdot \nabla s \, \delta M \right] = \alpha (\nabla \times \mathbf{F}) \cdot \nabla s \, \delta M. \tag{5.5}$$

The variability of $\delta M$ can now be related to sources and sinks of $s$, since clearly if fluid is entering or leaving through the ends of the volume, which lie on surfaces of constant $s$, then there must be sources or sinks of $s$.

The rate of mass flow across each end of the cylinder is

$$\rho \frac{ds}{dt} \frac{dn}{ds} \delta A,$$

so that the rate of change of mass in the cylinder is the convergence of the flux:

$$\frac{d}{dt} \delta M = -\frac{d}{dn} \left[ \rho \frac{ds}{dt} \frac{dn}{ds} \delta A \, \delta n \right] = -\rho \delta A \, \delta n \frac{\partial}{\partial s} \frac{ds}{dt} = -\delta M \frac{\partial}{\partial s} \frac{ds}{dt}. \tag{5.6}$$

Using this in (5.5) gives

$$\boxed{\begin{aligned} &\frac{d}{dt} \left[ \alpha (\nabla \times \mathbf{V} + 2\mathbf{\Omega}) \cdot \nabla s \right] = \\ &\quad \alpha (\nabla \times \mathbf{F}) \cdot \nabla s + \alpha (\nabla \times \mathbf{V} + 2\mathbf{\Omega}) \cdot \nabla \frac{SD}{dt}. \end{aligned}} \tag{5.7}$$

This known at *Ertel's Theorem* and states that the quantity

$$\alpha(\nabla \times \mathbf{V} + 2\mathbf{\Omega}) \cdot \nabla s$$

is conserved following the fluid flow, in the absence of friction or sources or sinks of $s$.

It is customary to use $\theta$ as the relevant state variable in the atmosphere, although it is more accurate to use $\theta_v$, since it accounts for the dependence of density of water vapor. We shall therefore define the *potential vorticity,* for atmospheric applications, as

$$q_a \equiv \alpha(\nabla \times \mathbf{V} + 2\mathbf{\Omega}) \cdot \nabla\theta_v, \tag{5.8}$$

and in the ocean, we will use potential density, $\sigma$, for $s$:

$$q_o \equiv \alpha(\nabla \times \mathbf{V} + 2\mathbf{\Omega}) \cdot \nabla\sigma. \tag{5.9}$$

Thus, according to (5.7), the conservation equations for $q_a$ and $q_o$ are

$$\frac{dq_a}{dt} = \alpha(\nabla \times \mathbf{F}) \cdot \nabla\theta_v + \alpha(\nabla \times \mathbf{V} + 2\mathbf{\Omega}) \cdot \nabla\frac{d\theta_v}{dt}, \tag{5.10}$$

and

$$\frac{dq_o}{dt} = \alpha(\nabla \times \mathbf{F}) \cdot \nabla\sigma + \alpha(\nabla \times \mathbf{V} + 2\mathbf{\Omega}) \cdot \frac{d\sigma}{dt}. \tag{5.11}$$

In the atmosphere, potential vorticity is conserved in the absence of friction or sources of $\theta_v$; it is conserved in the ocean in the absence of friction or sources of $\sigma$.

Returning to Figure 5.1, it is seen that in the absence of sources or sinks of $s$, the drawing together of the two $s$ surfaces implies, by mass conservation, that the volume expands laterally as it is squashed; by the Circulation Theorem, the absolute vorticity must decrease. This is precisely what (5.7) indicates. Potential vorticity can be thought of as that vorticity a fluid column would have if it were stretched or squashed to some reference depth.

*Volume conservation of potential vorticity*

The integral of potential vorticity over a finite mass of fluid is conserved even in the presence of friction or sources of $\sigma$ or $\theta_v$, as long as those effects vanish at the *boundaries* of that mass of fluid. The potential vorticity tendency integrated over a fixed (material) mass is

$$\int\int\int \frac{dq}{dt}\rho\, dx\, dy\, dz = \frac{d}{dt}\int\int\int q\rho\, dx\, dy\, dz.$$

We can take the time derivative outside the integral because mass is conserved. Using (5.7)

$$\frac{d}{dt}\int\int\int \rho q\, dx\, dy\, dz = \int\int\int \left[(\nabla \times \mathbf{F}) \cdot \nabla s + (\nabla \times \mathbf{V} + 2\mathbf{\Omega}) \cdot \nabla \frac{ds}{dt}\right] dx\, dy\, dz. \tag{5.12}$$

Since the divergence of the curl of any vector vanishes, and since $\mathbf{\Omega}$ is a constant vector, (5.12) can be rewritten

$$\frac{d}{dt}\int\int\int \rho q\,dx\,dy\,dz = \int\int\int \nabla\cdot\left[s(\nabla\times\mathbf{F}) + \frac{ds}{dt}(\nabla\times\mathbf{V}+2\boldsymbol{\Omega})\right]dx\,dy\,dz$$
$$= \int\int \left[s(\nabla\times\mathbf{F}) + \frac{ds}{dt}(\nabla\times\mathbf{V}+2\boldsymbol{\Omega})\right]\cdot\hat{n}\,dA, \tag{5.13}$$

where the last integral is over the entire surface bounding the volume and $\hat{n}$ is a unit vector normal to that surface. (We have used the divergence theorem here.) Thus, *the mass integral of $q$ is conserved if there are no sources of $s$ or friction on the boundaries of the fluid mass.*

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