Vertical structure and the WKB approximation

Vertical structure

In the presence of a basic state shear flow, $U(y,\xi)$, our QG equations for the waves become

$$(U-c)\left(\nabla^2\psi + \frac{1}{\overline{\rho}}\frac{\partial}{\partial\xi}\frac{\overline{\rho}f^2}{N^2}\frac{\partial}{\partial\xi}\psi\right) + \frac{\partial\overline{q}}{\partial y}\psi = 0$$

(the Rayleigh equation as extended) with

$$\frac{\partial\overline{q}}{\partial y} = \beta - \frac{\partial^2 U}{\partial y^2} - \frac{1}{\overline{\rho}}\frac{\partial}{\partial\xi}\frac{\overline{\rho}f^2}{N^2}\frac{\partial U}{\partial\xi}$$

The boundary equations are tricky, however. If U is not zero at the bottom, then we will have pressure gradients along the surface so the the surface coordinate, $\overline{\xi}_s(y)$ will be a function of y. The condition that the bottom be at geopotential 0 becomes

$$\overline{\phi}(y,\overline{\xi}_s) = 0$$

Taking a y derivative and using the geostrophic relation gives

$$-fU_s + \overline{b}_s\frac{\partial\overline{\xi}_s}{\partial y} = 0$$

where U_s and $\overline{b}_s$ are the surface values of velocity and buoyancy. The linearized boundary conditions become

$$\overline{b}_s\xi_s' + \phi'(x,y,\overline{\xi}_s,t) = 0$$

$$\left(\frac{\partial}{\partial t} + U_s\frac{\partial}{\partial x}\right)\xi_s' + v'\frac{\partial\overline{\xi}_s}{\partial y} = \omega'$$

Multiplying the second equation by $-\overline{b}_s$ and substituting from the first (and the equation for the basic gradient of the surface coord.) gives

$$\left(\frac{\partial}{\partial t} + U_s\frac{\partial}{\partial x}\right)\phi' - v'fU_s = -\overline{b}_s\omega'$$

We combine this with the buoyancy equation

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\frac{\partial\phi'}{\partial\xi} - v'f\frac{\partial U}{\partial\xi} = -N^2\omega'$$

To get

$$\left(\frac{\partial}{\partial t} + U_s\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial\xi} - \frac{N^2}{\overline{b}_s}\right)\phi' - v'f\left(\frac{\partial}{\partial\xi} - \frac{N^2}{\overline{b}_s}\right)U = 0 \quad at \quad \xi = \overline{\xi}_s$$

Simple modes

For the purpose of examining simple modes, we shall assume the surface velocity U_s vanishes. Then we can take $\overline{\xi}_s = 0$. We also take the vertical shear to be zero at the ground so that the surface density is constant and $\overline{b}_s = g$. Our boundary condition is now

$$\frac{\partial}{\partial t}\left(\frac{\partial}{\partial \xi} - \frac{N^2}{g}\right)\phi' = 0 \quad at \quad \xi = 0$$

$$\Rightarrow \quad \frac{\partial}{\partial \xi}\phi' = \frac{N^2}{g}\phi' \quad at \quad \xi = 0$$

We shall take the isothermal basic state so that

$$\frac{1}{\overline{\rho}}\frac{\partial}{\partial \xi}\frac{\overline{\rho} f^2}{N^2}\frac{\partial}{\partial \xi} \longrightarrow \frac{f^2}{N^2}\left(\frac{\partial^2}{\partial \xi^2} - \frac{1}{H_s}\frac{\partial}{\partial \xi}\right)$$

and look for solutions with $\psi = e^{\xi/2H_s}\Psi$ which implies

$$\left(\frac{\partial^2}{\partial \xi^2} - \frac{1}{H_s}\frac{\partial}{\partial \xi}\right)\psi = e^{\xi/2H_s}\left(\frac{\partial^2}{\partial \xi^2} - \frac{1}{4H_s^2}\right)\Psi$$

Our interior equation simplifies to

$$\frac{\partial^2}{\partial \xi^2}\Psi = -\Gamma(\xi)\Psi \qquad (1)$$

under the assumption that c is less than the minimum of U (or greater than the maximum) with

$$\Gamma(\xi) = \frac{N^2}{f^2}\frac{\frac{\partial \overline{q}}{\partial y}}{U - c} - \frac{N^2}{f^2}k^2 - \frac{1}{4H_s^2}$$

The lower boundary condition is

$$\frac{\partial}{\partial \xi}\Psi = -\kappa\Psi \qquad (2)$$

with

$$\kappa = \frac{1}{2H_s} - \frac{N^2}{g} = \frac{1}{H_s}\left(\frac{1}{2} - \frac{R}{c_p}\right)$$

No flow

In the absence of flow, we have

$$\Psi = \exp(-\kappa\xi) \quad and \quad c = -\frac{\beta}{k^2 + \frac{f^2}{N^2 H_s^2}\frac{R}{c_p}(1 - \frac{R}{c_p})}$$

or

$$c = -\frac{\beta}{k^2 + \frac{f^2}{gH_s}\frac{c_v}{c_p}} = -\frac{\beta}{k^2 + \frac{f^2}{gH_e}} \quad , \quad H_e = H_s\frac{c_p}{c_v}$$

Flow

In the presence of flow can we have more than one mode? Given a $U(\xi)$ profiles, we can integrate (1) starting from the initial conditions $\Psi(0) = 1$, $\frac{\partial}{\partial \xi}\Psi = -\kappa$ and vary c until the solution decays at ∞, With

$$U = \frac{s}{2}\xi^2$$

we have

$$\Gamma(\xi) = \frac{\frac{N^2}{f^2}\beta - s + \dfrac{s\xi}{H_s}}{\frac{1}{2}s\xi^2 - c} - \frac{N^2}{f^2}k^2 - \frac{1}{4H_s^2}$$

For $s < N^2\beta/f^2$, we can take $c < 0$ so that the first term is positive but becomes small in the upper atmosphere, while the second and third terms are negative. Thus we have sinusoidal solutions in the lower atmosphere where $\Gamma(\xi)$ is positive and exponentially decays solutions at height where Γ becomes negative. As long as the sinusoidal range is big enough, we should be able to find several modes.

WKB

We can look first for trapped mode solutions to (1) with $\Gamma > 0$ everywhere using a WKB form. We let

$$\Psi = A(\xi)\exp(-\theta(\xi))$$

giving

$$A'' + A\theta'^2 - 2A'\theta' - A\theta'' = |\Gamma|A$$

Taking the presumed largest terms to balance gives

$$\theta = \int_0^\xi |\Gamma(\xi')|^{1/2} d\xi'$$

The next order terms give us the structure of A, but that isn'r really necessary here. The lower boundary condition gives, to the first approximation,

$$-\theta'(0)A = -\kappa A \quad \Rightarrow \quad |\Gamma(0)|^{1/2} = \kappa \quad \textit{or} \quad \Gamma(0) = -\kappa^2$$

so that

$$c = -\frac{\beta - \dfrac{f^2}{N^2}s}{k^2 + \dfrac{f^2}{4N^2H_s^2} - \dfrac{f^2}{N^2}\kappa^2}$$

To solve (1) approximately using WKB for the other modes, we must deal with the turning point at $\xi = \xi_0$ where $\Gamma(\xi_0) = 0$ (although we don't know where exactly that is until we determine c). Below ξ_0, we can take

$$\Psi = A(\xi)\cos\theta(\xi)$$

giving

$$A'' - A\theta'^2 = -\Gamma A$$
$$-2A'\theta' - A\theta'' = 0$$

The WKB approximation involves dropping the A'' term; this gives

$$\theta = \theta_0 + \int_0^\xi \Gamma^{1/2}$$

and

$$A = \left(\frac{\Gamma(0)}{\Gamma}\right)^{1/4}$$

The lower boundary condition to lowest order gives

$$\tan\theta_0 = \frac{\kappa}{\sqrt{\Gamma(0)}} \tag{3}$$

As we approach the turning point, A blows up (but very slowly); the solution looks like

$$\Psi = \frac{\Gamma(0)^{1/4}}{\Gamma(\xi)^{1/4}} \cos\left(\theta_0 + \int_0^{\xi_0} \Gamma^{1/2} + \int_{\xi_0}^{\xi} \Gamma^{1/2}\right)$$

Expanding the last term using $\Gamma(\xi) \sim -|\Gamma'(\xi_0)|(\xi - \xi_0)$ gives

$$\Psi \sim \frac{\Gamma(0)^{1/4}}{\Gamma(\xi)^{1/4}} \cos\left(\theta_0 + \int_0^{\xi_0} \Gamma^{1/2} - \frac{2}{3}|\Gamma'(\xi_0)|^{1/2}(\xi_0 - \xi)^{3/2}\right)$$

But the solution isn't singular at the turning point; instead, we must look for solutions in the vicinity of ξ_0 and use these to match between the solution below and the decaying solutions (like those considered previously) above. In the vicinity of the turning point, the equation looks approximately like

$$\frac{\partial^2}{\partial\xi^2}\Psi \simeq |\Gamma'(\xi_0)|(\xi - \xi_0)\Psi$$

The solutions to this are Airy functions which are well-behaved at the turning point, become sinusoidal for $\xi < \xi_0$ and decay exponentially for $\xi > \xi_0$:

$$\Psi = \alpha \ \mathrm{Ai}\left(|\Gamma'(\xi_0)|^{1/3}[\xi - \xi_0]\right)$$

The asymptotic form of this is

$$\Psi \to \alpha\pi^{-1/2}|\Gamma'(\xi_0)(z_0 - z)^{-1/4} \cos\left(\frac{\pi}{4} - \frac{2}{3}|\Gamma'(\xi_0)|^{1/2}(\xi_0 - \xi)^{3/2}\right)$$

Matching these gives us a second condition

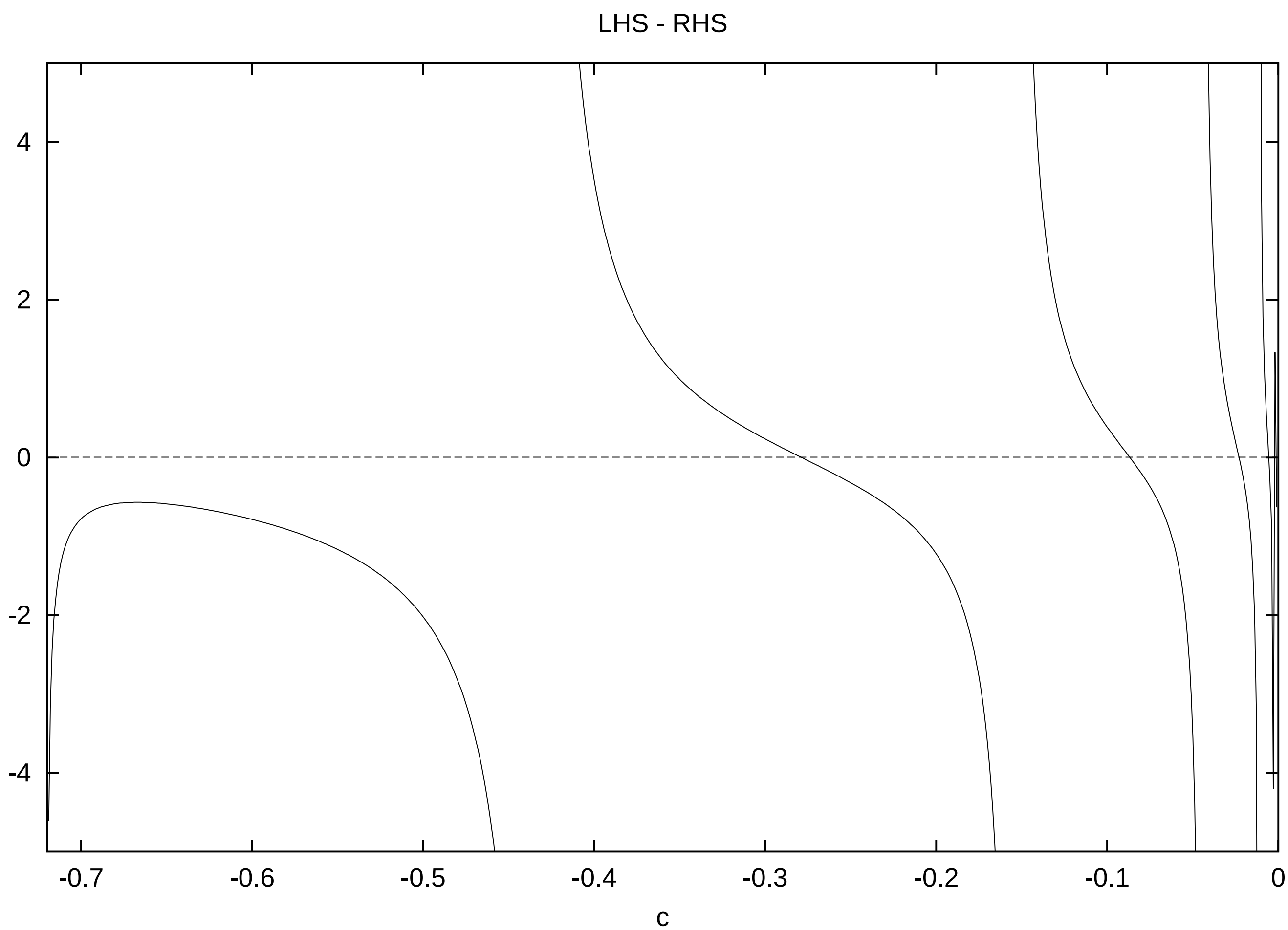
$$\theta_0 + \int_0^{\xi_0} \Gamma^{1/2} = \frac{\pi}{4} \tag{4}$$

Combining equations (3) and (4) gives an implicit expression for c

$$\tan(\frac{\pi}{4} - \int_0^{\xi_0} \Gamma^{1/2}) = \frac{\kappa}{\sqrt{\Gamma(0)}} \tag{5}$$

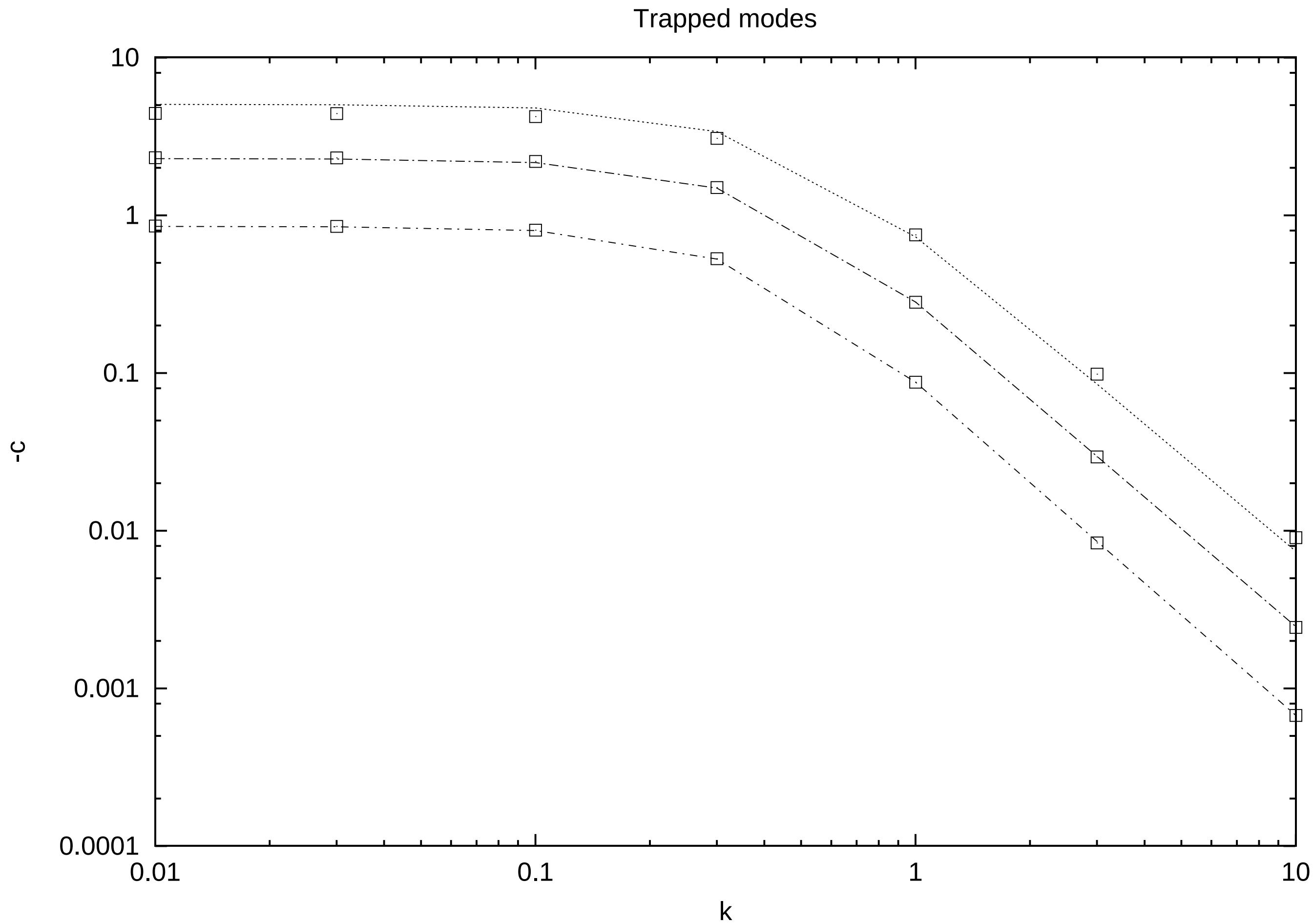
I.e., given k and a guess of c, we know Γ and hence ξ_0. Evaluating the integral and substituting in (5) gives us a test function; we can vary c until it's zero.

The following figure shows the difference between the l.h.s. and r.h.s. of (5) as $c/\beta\frac{H^2 H_s^2}{f^2}$ varies for a given $kH_sN/f = 1$ and $s/\beta\frac{N^2}{f^2} = 0.1$ We can then search in the vicinities of the zeros to find the speeds.



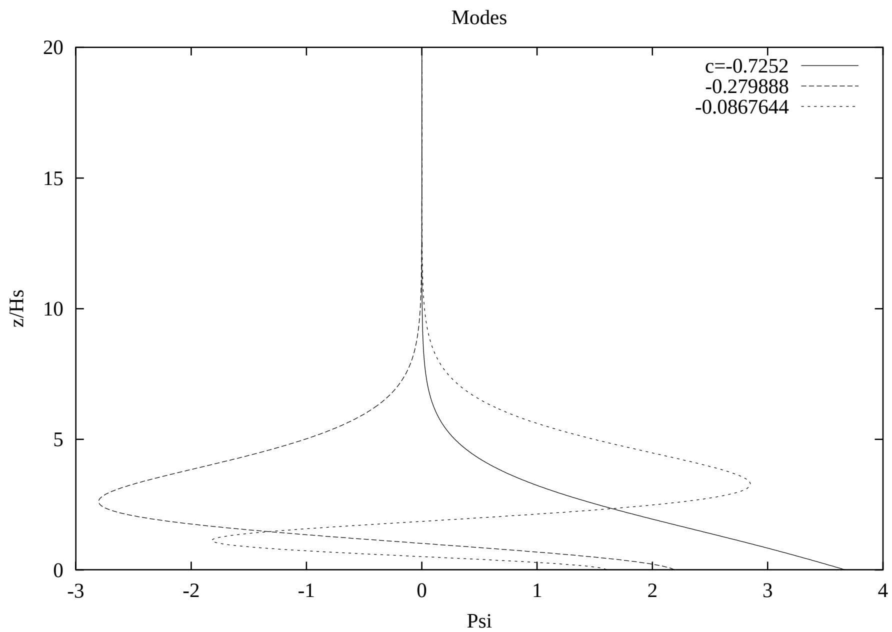
Mismatch in (5) vs. c scaled by the characteristic Rossby wave speed $\beta N^2 H_s^2/f^2$. The eigenvalues are where c values for which the function is zero.

The next figure compares the speeds from the WKB analysis and from a direct shooting-method solution to (1) and (2). The WKB analysis (trading off numerical for analytical complexity!) works very well.

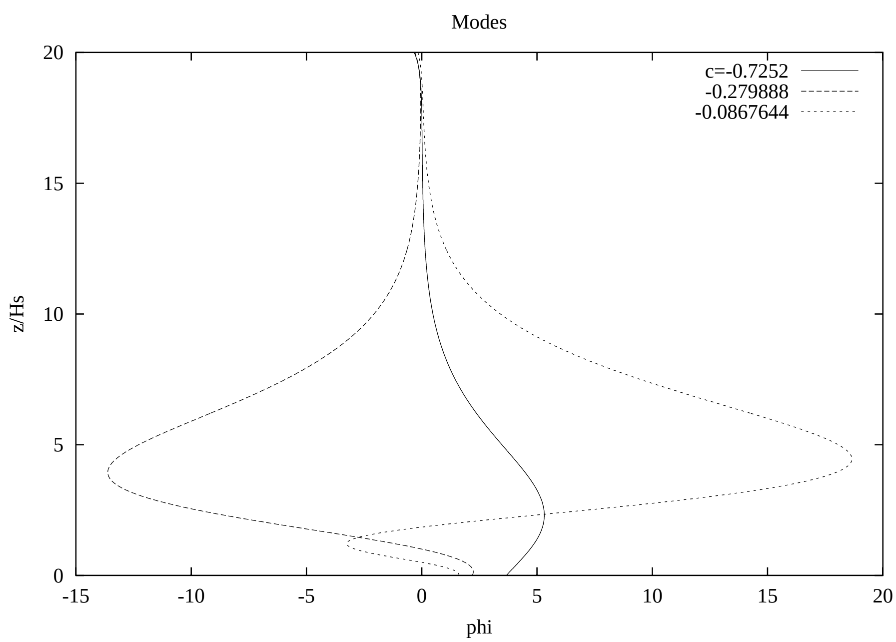


Numerical (solid) and WKB (boxes) estimates of phase speeds for first three modes (the gravest being the one which propagates most rapidly).

The last figure shows the first three modes from the numerical solution for the parameters above.



Modal structure $\Psi(\xi)$ for the first three modes at $kH_sN/f = 1$ and $s/\beta\frac{N^2}{f^2} = 0.1$



Modal structure $\phi(\xi) = \Psi(\xi)\exp(\xi/2H_s)$