9. The oceanic waveguide – normal modes of a stratified, rotating fluid

So far we have considered internal gravity waves in an unbounded domain. We now consider the fluid bounded by a flat bottom at z = -D and a free surface η around the rest position z = 0.

We now need boundary conditions at z = -D and z = η while the equations of motion in the interior remain the same leading to the final master equation in the Boussinesq approximation

$$\frac{1}{\rho_o}\frac{\partial \rho_o}{\partial z} << \frac{\partial^2 w/\partial z^2}{\partial w/\partial z} \qquad \text{or} \qquad \frac{1}{\rho_o}\frac{\partial \rho_o}{\partial z}\frac{\partial w}{\partial z} / \frac{\partial^2 w}{\partial z^2} << 1$$

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)+f^2\frac{\partial^2 w}{\partial z^2}+N\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)=0$$

At z = -D w=o simple bottom b.c.

At z = η

$$w = \frac{\partial \eta}{\partial t} \qquad \text{linearized kinematic b,c}$$

$$p(x,y,\eta) = o \qquad \text{unforced waves}$$

Let us expand p around z = o as we have small (linear) motions

$$p(x,y,\eta) = p(x,y,o) + \frac{\partial p}{\partial z}\Big|_{z=o}\ \eta$$

In the momentum equations we assumed a basic state (ρ_o,p_o) for which

$\rho_{total} = \rho_o + \rho$; $p_{total} = p_o + p$ and $\frac{\partial p_o}{\partial z} = -\rho_o g$. Then the basic state cancels out in the third momentum equations which gives

$$\frac{\partial p}{\partial z} = -\rho_o \frac{\partial w}{\partial t} - \rho g$$

However in the surface boundary condition we have

$p = p_{total} = p_o(z) + p$.

So $\frac{\partial p_{total}}{\partial z} = \frac{\partial p_o}{\partial z} + \frac{\partial p}{\partial z}$ where p is the perturbation pressure. Then:

$$\frac{\partial p}{\partial z} = -\rho g - \rho_o \frac{\partial w}{\partial t}$$

and $$\frac{\partial p_{total}}{\partial z} = -(\rho_o + \rho)g - \rho_o \frac{\partial w}{\partial t}$$

The first term in the expansion is $\frac{\partial p}{\partial z}|_{z=o}\ \eta = o(a^2)$

If we consider the terms $\left(\rho g, \rho_o \frac{\partial w}{\partial t}\right)$ we obtain terms $o(a^2)$

$$(g\rho\eta;\ \rho_o \frac{\partial w}{\partial t}\eta) = o(a^2)$$

Therefore, to be consistent in the linearization, we keep only $(-\rho_o g)$

$p(x,y,\eta) = p(x,y,o) - \rho_o g\eta = o$ at $z = o$

Now combine $p(x,y,o) = \rho_o g\,\eta$

at $z = o$

$$\frac{\partial \eta}{\partial t} = w$$

$$\frac{\partial}{\partial t} p(x,y,o) = \rho_o g w \quad \text{at} \quad z = o$$

Take the horizontal Laplacian of the above to eliminate the perturbation pressure.

$$\frac{\partial}{\partial t}\nabla_H^2 p = \rho_o g \nabla_H^2 w \quad \text{at} \quad z = o$$

But from Eq. III of the unbounded rotation case

$$\frac{1}{\rho_o}\frac{\partial}{\partial t}\nabla_H^2 p = \left[\frac{\partial^2}{\partial t^2} + f^2\right]\frac{\partial w}{\partial z} \qquad \text{everywhere hence also at } z = 0$$

and

$$\left[\frac{\partial^2}{\partial t^2} + f^2\right]\frac{\partial w}{\partial z} = g\nabla_H^2 w \qquad \text{at } z = o$$

Therefore, in terms of w, we have

$$\left(\frac{\partial^2}{\partial t^2} + f^2\right)\frac{\partial w}{\partial z} - g\nabla_H^2 w = o \qquad \text{at } z = o$$

$$\frac{\partial^2}{\partial t^2}\nabla^2 w + f^2\frac{\partial^2 w}{\partial z^2} + N^2\nabla_H^2 w = o \quad \text{in the interior}$$

$$w = o \ \text{ at } \ z = -D$$

Let us restrict ourselves to N^2 = constant and orient the x-axis in the direction of K_H. So we look for a solution of the form

$$w = W(z)e^{i(kx-\omega t)}$$

The problem becomes:

$$(\omega^2 - f^2)W_z - gk^2 W = 0 \quad \text{at } z = o$$

$$W_{zz} + k^2\left(\frac{N^2 - \omega^2}{\omega^2 - f^2}\right)W = o \quad \text{in the interior}$$

$$W = o \quad \text{at } z = -D$$

Consider the two quantities:

$$S^2 = \omega^2 - f^2 \qquad R^2 = \frac{N^2 - \omega^2}{\omega^2 - f^2}$$

For the realistic case $N^2 > f^2$, we have the following cases to consider

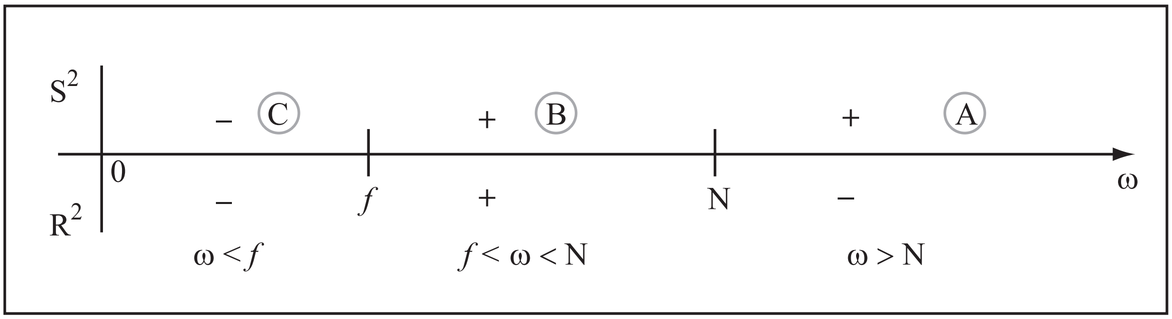


Figure by MIT OpenCourseWare.

Case C

$$S^2 = \omega^2 - f^2 < o \qquad\qquad S_1^2 = f^2 - \omega^2 > 0$$

Define:

$$R^2 = \frac{N^2 - \omega^2}{\omega^2 - f^2} \qquad\qquad R_1^2 = \frac{N^2 - \omega^2}{f^2 - \omega^2}$$

The problem becomes:

$$S_1^2 W_z + gk^2 W = 0 \quad \text{at } z = o$$

$$W_{zz} - k^2 R_1^2 W = o \quad \text{interior}$$

$$W = o \quad \text{at } z = -D$$

The solution to the interior problem is $W = e^{\pm kR_1(z+\alpha)}$. We can consider hyperbolic sines and cosines and the solution which satisfies the bottom b.c. is

$$W = \sinh[kR_1(z+D)]$$

So $\quad w = e^{i(kx-\omega t)} \sinh[kR_1(z+D)]$

Substituting into the surface b.c. we obtain the dispersion relation:

$$S_1^2 kR_1\cosh(kR_1D) + gk^2\sinh(kR_1D) = 0$$

$$\frac{S_1^2R_1}{gk} = \frac{R_1(f^2-\omega^2)}{gk} = -\tanh(kR_1D)$$

We can solve the problem graphically

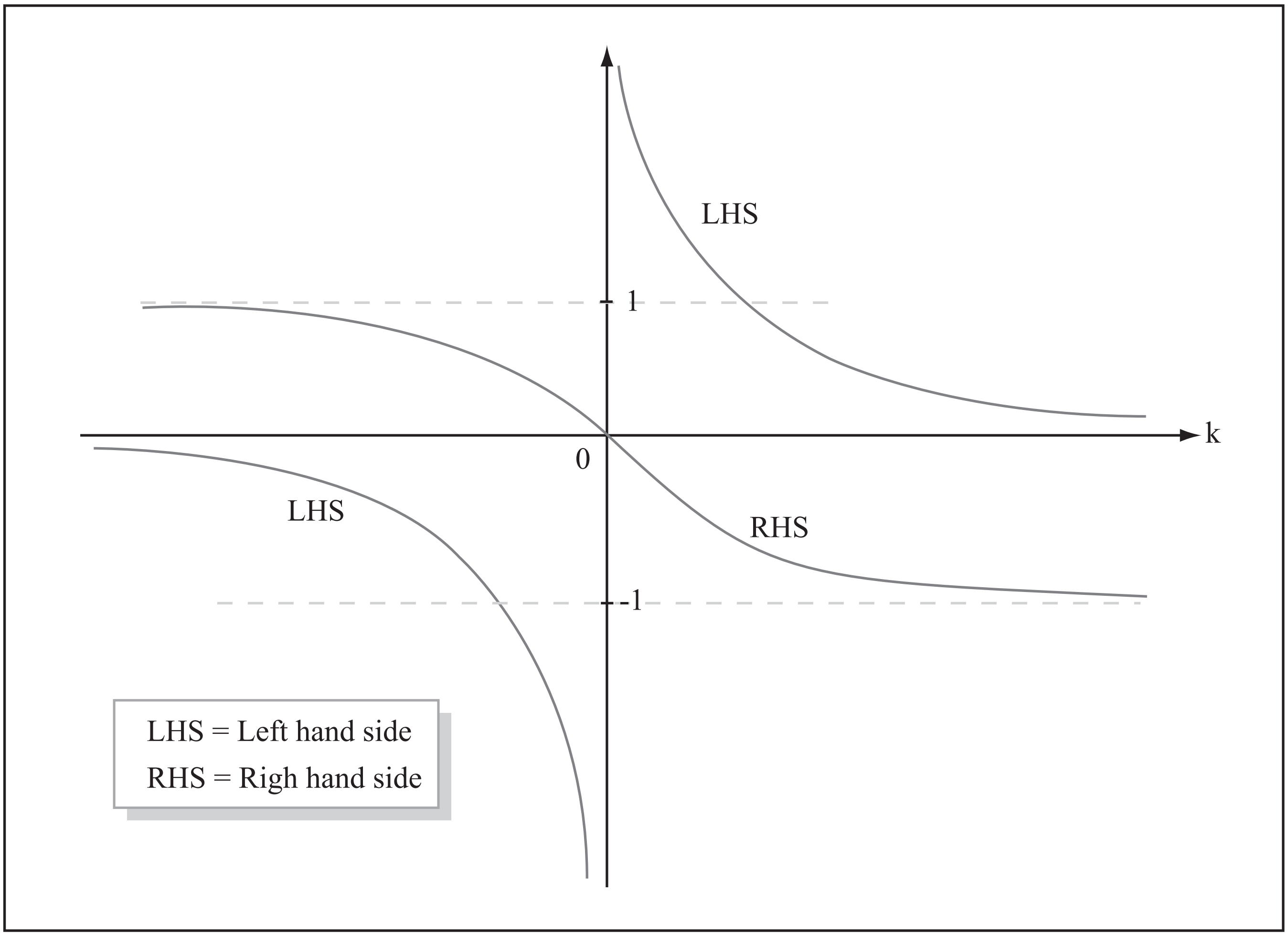


Figure by MIT OpenCourseWare.

No wave exists as solution.

<u>Case A</u>

$$S^2 = \omega^2 - f^2 > 0$$

$$R^2 = \frac{N^2-\omega^2}{\omega^2-f^2} < 0 \qquad \text{Again define} \quad R_1^2 = \frac{\omega^2-N^2}{\omega^2-f^2} > 0$$

Then the problem is:

$$(\omega^2\text{-}f^2)W_z\text{-}gk^2W = 0 \quad \text{at} \quad z\text{=}o$$

$$W_{zz} - k^2 R_1 W = 0 \qquad \text{interior}$$

$W = o$ at $z=-D$

Like in case C, the solution that satisfies the bottom boundary condition is:

$W = \sinh[kR_1(z+D)]$ and $w = e^{i(kx-\omega t)} \sinh[kR_1(z+D)]$

But now the dispersion relationship is

$$(\omega^2 - f^2)kR_1 - \cosh[kR_1 D] - gk\sinh[kR_1 D] = 0$$

$$\frac{R_1(\omega^2 - f^2)}{gk} = \frac{R_1 S^2}{gK} = +\tanh(kR_1 D)$$

Again we solve graphically

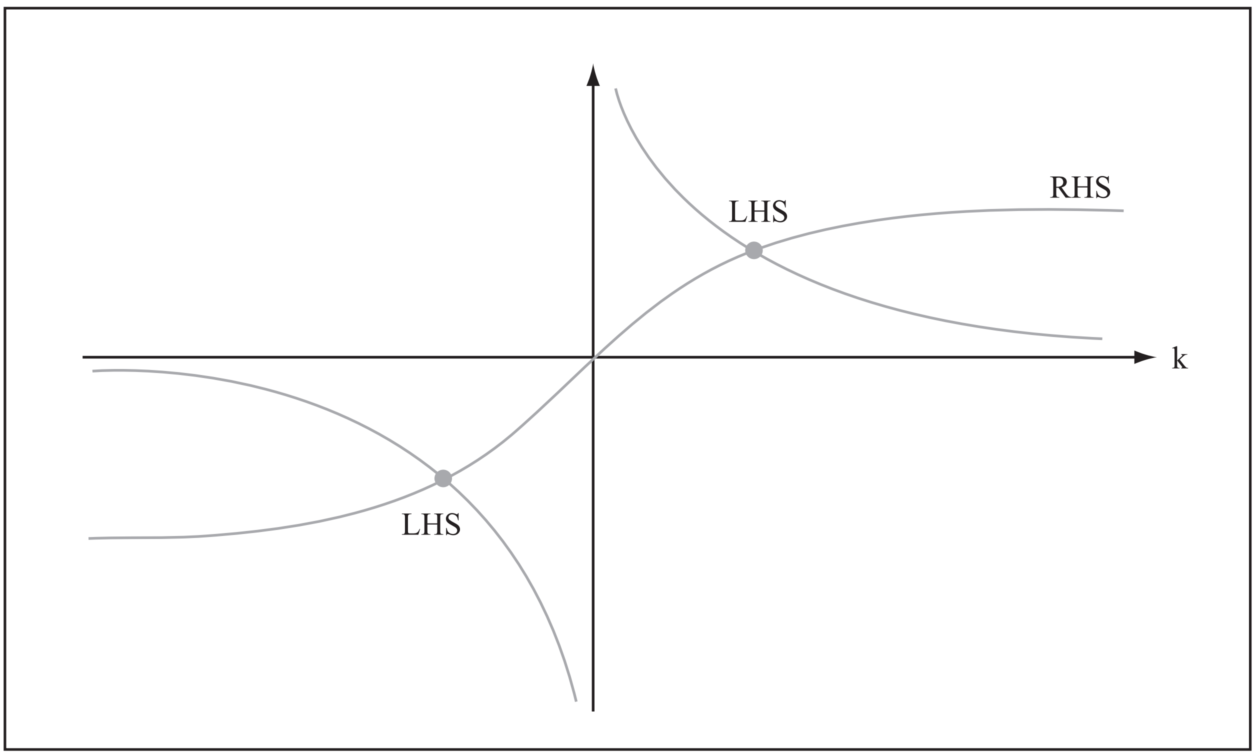


Figure by MIT OpenCourseWare.

There are two solutions, two oppositely traveling waves. Write the dispersion relationship as

$$\omega^2 = f^2 + \frac{gk}{R_1} \tanh(kR_1 D).$$

$$R_1^2 = \frac{\omega^2 - N^2}{\omega^2 - f^2} \quad \text{if} \quad \omega >> (N, f) \rightarrow R_1 \simeq 1$$

$$\omega^2 \simeq f^2 + gk\tanh(kD)$$

These are surface gravity waves modified by rotation.

Case B

The most interesting case corresponding to the realistic range

$$f<\omega<N$$

Then

$$S^2 = \omega^2 - f^2 > o \qquad R^2 = \frac{N^2 - \omega^2}{\omega^2 - f^2} > 0$$

The problem is:

$$(w^2-f^2)W_z - gk^2W = 0 \ \text{ at } z = o$$

$$W_{zz} + k^2R^2W = 0$$

$$W = o \ \text{ at } \ z = -D$$

The solutions will be oscillatory $e^{\pm imz} = e^{\pm kRz}$

And the solution satisfying the bottom b.c is

$$W = \sin[kR(z+D)] \qquad w = e^{i(kx-\omega t)}\sin[kR(z+D)]$$

The vertical wave number is

$$m^2 = k^2R^2 = k^2\left(\frac{N^2-\omega^2}{\omega^2-f^2}\right) \to k^2\left(\frac{N^2}{\omega^2}-1\right) \quad \text{if} \quad f = o$$

The surface b,c gives

$$(\omega^2-f^2)m\cos(mD) - gk^2\sin(mD) = o \qquad m=kR$$

$$\text{or} \quad (\omega^2-f^2)R\cos(kRD) - gk\sin(kRD) = o$$

$$\frac{R(\omega^2-f^2)}{gk} = \tan(kRD)$$

is the dispersion relation, solved graphically.

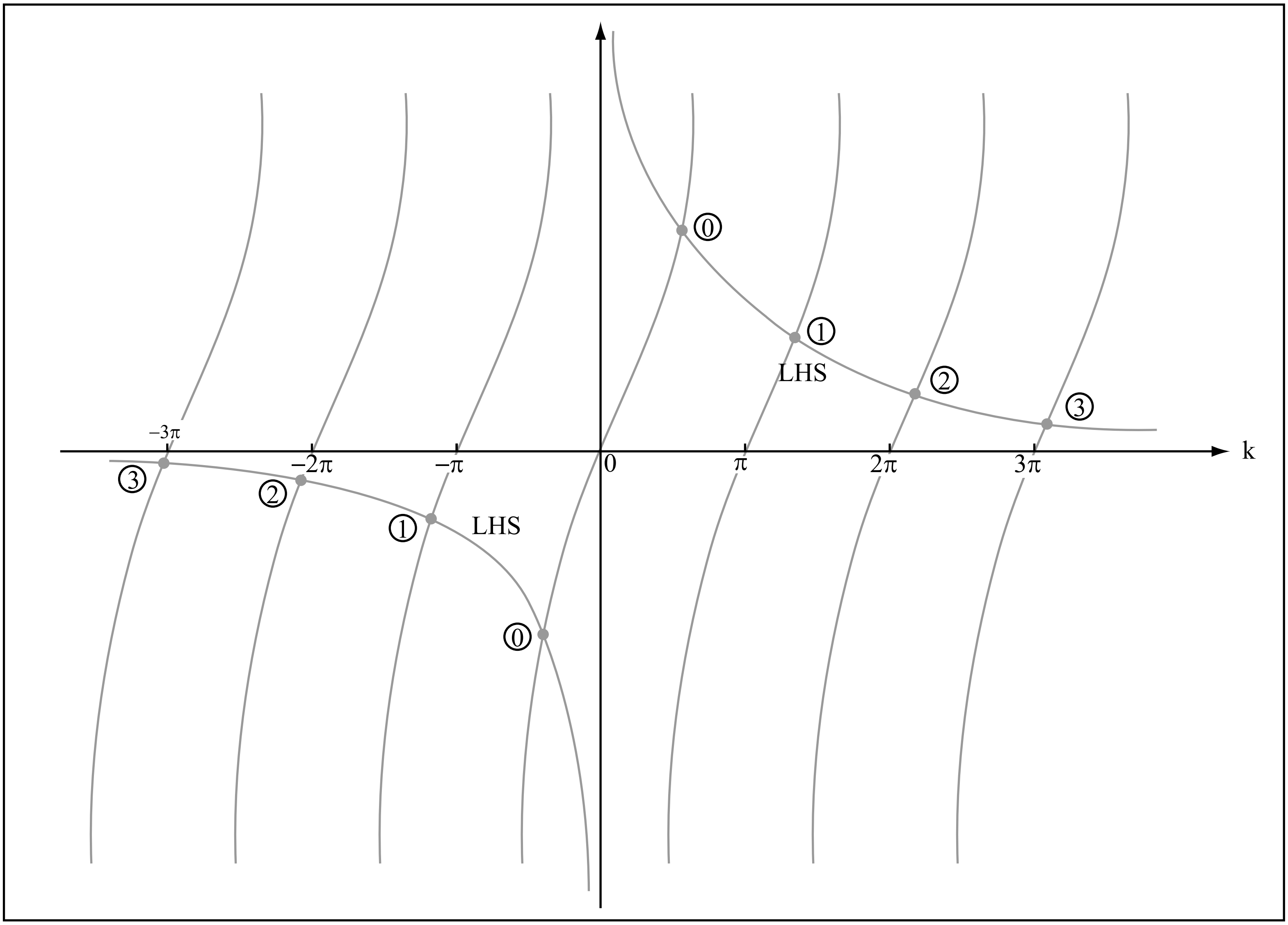


Figure by MIT OpenCourseWare.

The solution is quantized, the wave numbers are (m_n, k_n) with n = 0,±1,±2,±3 . . .

There is an infinite, denumerable number of solutions traveling in opposite directions.

The 0 mode has kRD<<1=>tan(kRD)≈kRD

$$\frac{R(\omega^2 - f^2)}{gk_o} \simeq k_o RD$$

$$\frac{\omega^2 - f^2}{gD} = k_o^2 \quad \rightarrow \quad \omega^2 = k_o^2(gD) + f^2$$

The zero mode is a surface gravity wave in shallow water modified by rotation.

For large k, the intersection of the two curves are very near ($n\pi$). So

$$k_n RD \simeq \pm n\pi$$

and we can use this as the dispersion relation.

Notice that if we require a rigid list as surface b,c, then the dispersion relation is

$$\sin(k_n RD) = 0$$

and $k_n RD = \pm n\pi$ becomes exact. But we lose the surface gravity mode which requires a free surface η to exist. The above, however, tells us that for the internal gravity waves the free surface acts as if it were rigid, and the eigensolutions are those which can be found imposing $w = 0$ at $z = 0$. For the internal modes, $k_n RD \sim \pm n\pi$ gives

$$k_n D\left(\frac{N^2 - \omega^2}{\omega^2 - f^2}\right)^{1/2} \simeq \pm n\pi$$

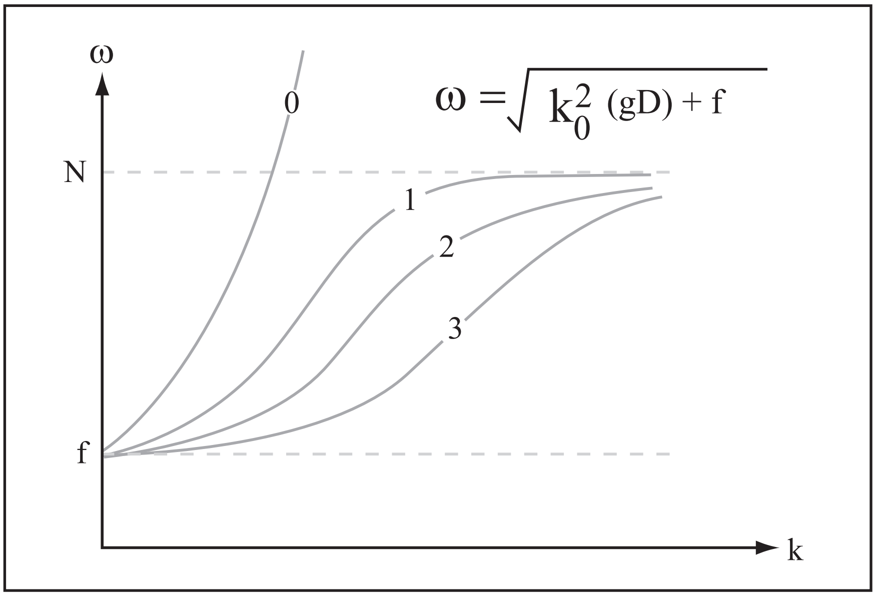
Hence

$$(\omega^2 - f^2)\left(\frac{n\pi}{k_n D}\right)^2 \sim (N^2 - \omega^2)$$

$$\omega^2\left(\frac{n\pi}{k_n D}\right)^2 + \omega^2 = N^2 + f^2\left(\frac{n\pi}{k_n D}\right)^2$$

$$\omega^2\left[1 + \left(\frac{n\pi}{k_n D}\right)\right]^2 = N^2 + f^2\left(\frac{n\pi}{k_n D}\right)^2$$

So, for every k_n,



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From the previous version of the dispersion relationship

$$k_n D(N^2-\omega^2)^{1/2} = n\pi(\omega^2-f^2)^{1/2} = 0$$

$\omega =10f$ is the lower limit, and $k_n = 0$.

For all curves, if $k_n \to \infty \quad \omega \to N$

N is the upper limit for ω and the curves reach it asymptotically.

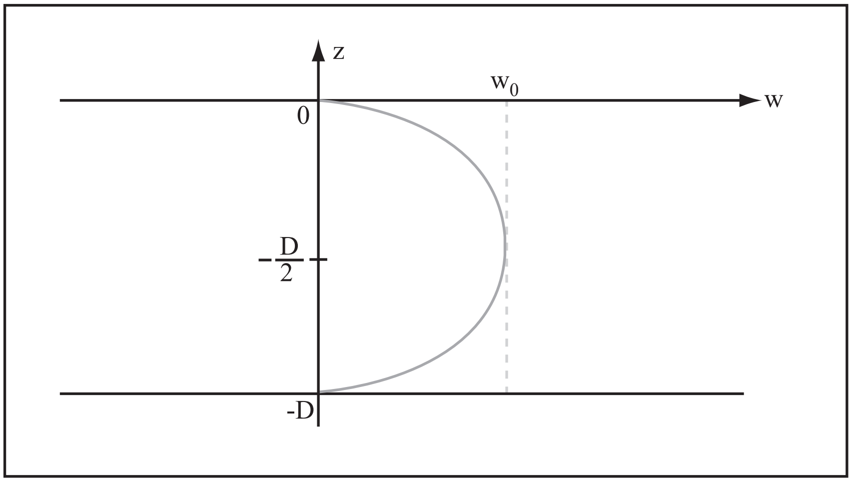
The two limiting points are those when

$$c_g = \frac{\partial\omega}{\partial k} = 0$$

Consider the lowest mode n = 1.

From the dispersion relation $k_1 = \dfrac{\pi}{RD}$

If $\quad w = w_o \cos(k_1 x - \omega t)\sin[\frac{\pi}{D}(z + D)]$ (real part)



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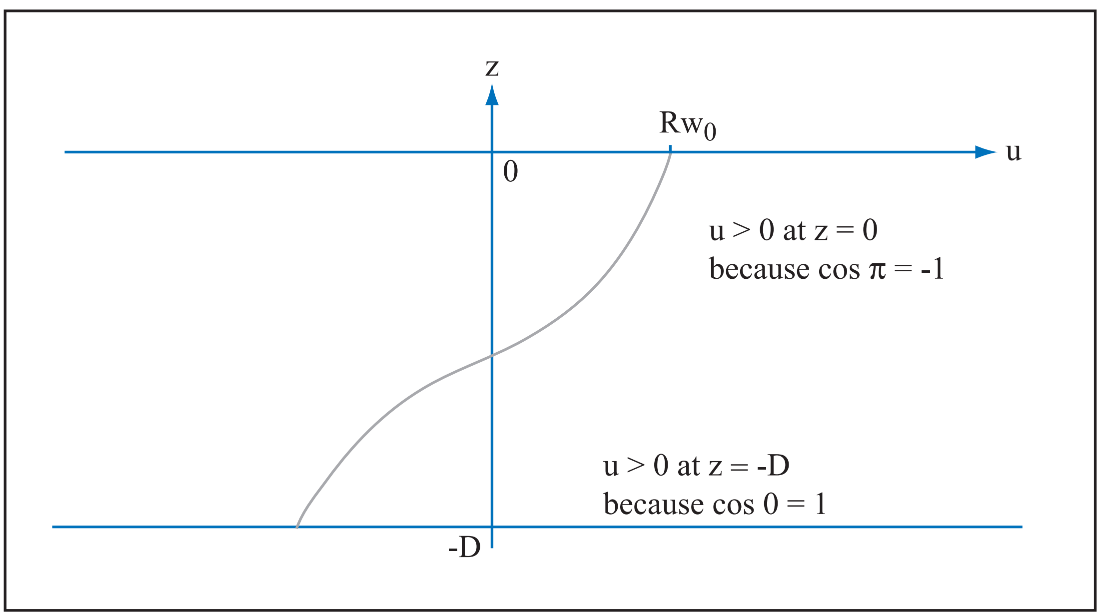
From $u_x + w_z = 0$

$$u = -Rw_o \sin(k_1 x - \omega t)\cos[\frac{\pi}{D}(z + D)]$$



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Consider a snapshot of the wave at time t* as a function of $k_1 x - \omega t^* = \frac{\pi x}{RD} - \omega t^*$.

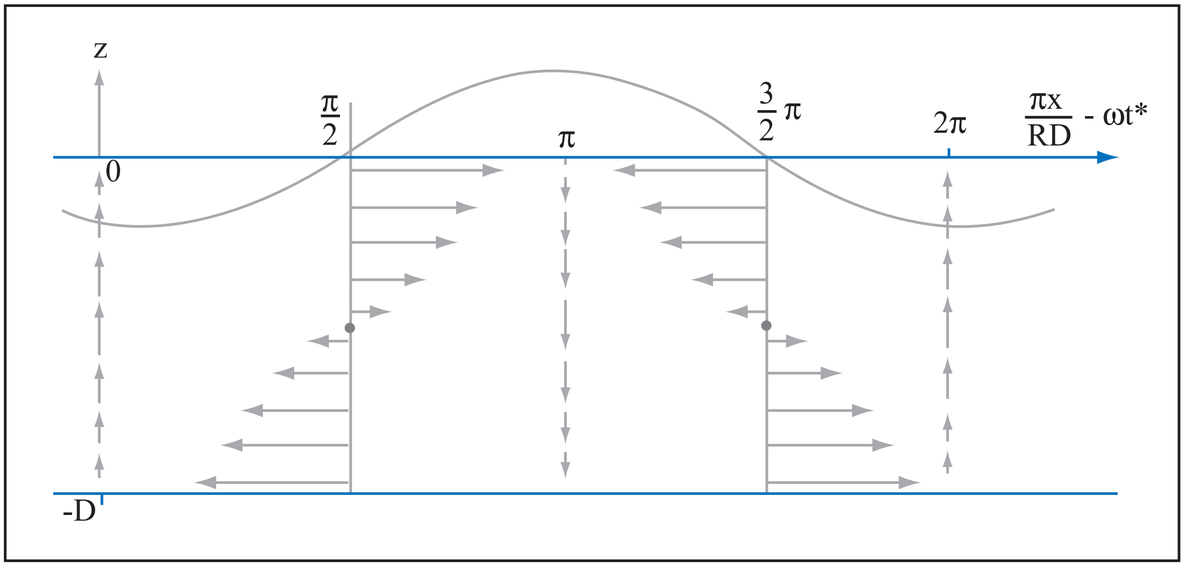


Figure by MIT OpenCourseWare.

$$\text{At } \left(\frac{\pi x}{RD} - \omega t^*\right) = 0 \quad u = o \quad w = w_o \sin\left[\frac{\pi}{D}(z + D)\right] \;\; > o$$

$$\text{At } \left(\frac{\pi x}{RD} - \omega t^*\right) = \frac{\pi}{2} \quad u = -Rw_o \cos\left[\frac{\pi}{D}(z + D)\right]\begin{matrix} > o \;\text{ at }\; z - o \\ < o \;\text{ at }\; z = -D \end{matrix}; \;\; w = o$$

$$\text{At } \left(\frac{\pi x}{RD} - \omega t^*\right) = \pi \quad u = o; \quad w = -w_o \sin\left[\frac{\pi}{D}(z + D)\right] \;\; < o$$

$$\text{At } \left(\frac{\pi x}{RD} - \omega t^*\right) = \frac{3}{2}\pi \quad u = +Rw_o \cos\left[\frac{\pi}{D}(z + D)\right]\begin{matrix} < o \;\text{ at }\; z = o \\ > o \;\text{ at }\; z = -D \end{matrix}; \;\; w = o$$

The particle motions consist of a series of convergences under the crests (downwelling) and divergences under the troughs (upwelling) of the travelling wave -> system of CELLS.

<u>Variable $N^2(z)$</u>

The more realistic situation is the one sketched in the figure - then

$$R^2(z) = \frac{N^2(z) - \omega^2}{\omega^2 - f^2}$$

We again seek a solution of the form

$$w = e^{i(kx-\omega t)}\ W)z)$$

and the problem is the same formally:

$$(\omega^2 - f^2)W_z - gk^2W = 0 \quad \text{at } z=o$$

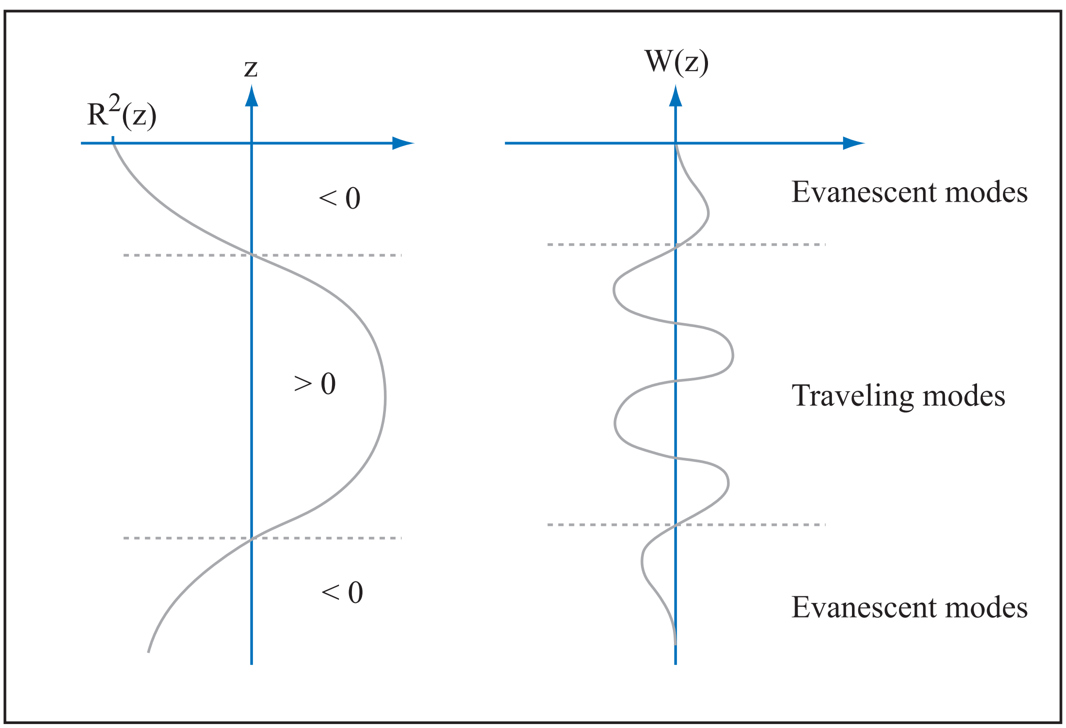
$$W_{zz} + K^2R^2(z)W = o \quad \text{interior}$$

$$W = o \quad \text{at } z = -D$$

If $R^2(z)>o$, then the solution is a traveling wave e^{imz}

If $R^2(z)<o$, the solution is an evanescent mode e^{-mz}

For a wave traveling in x we have the following situation, supposing the $R^2(z)$ profile is as in the sketch.



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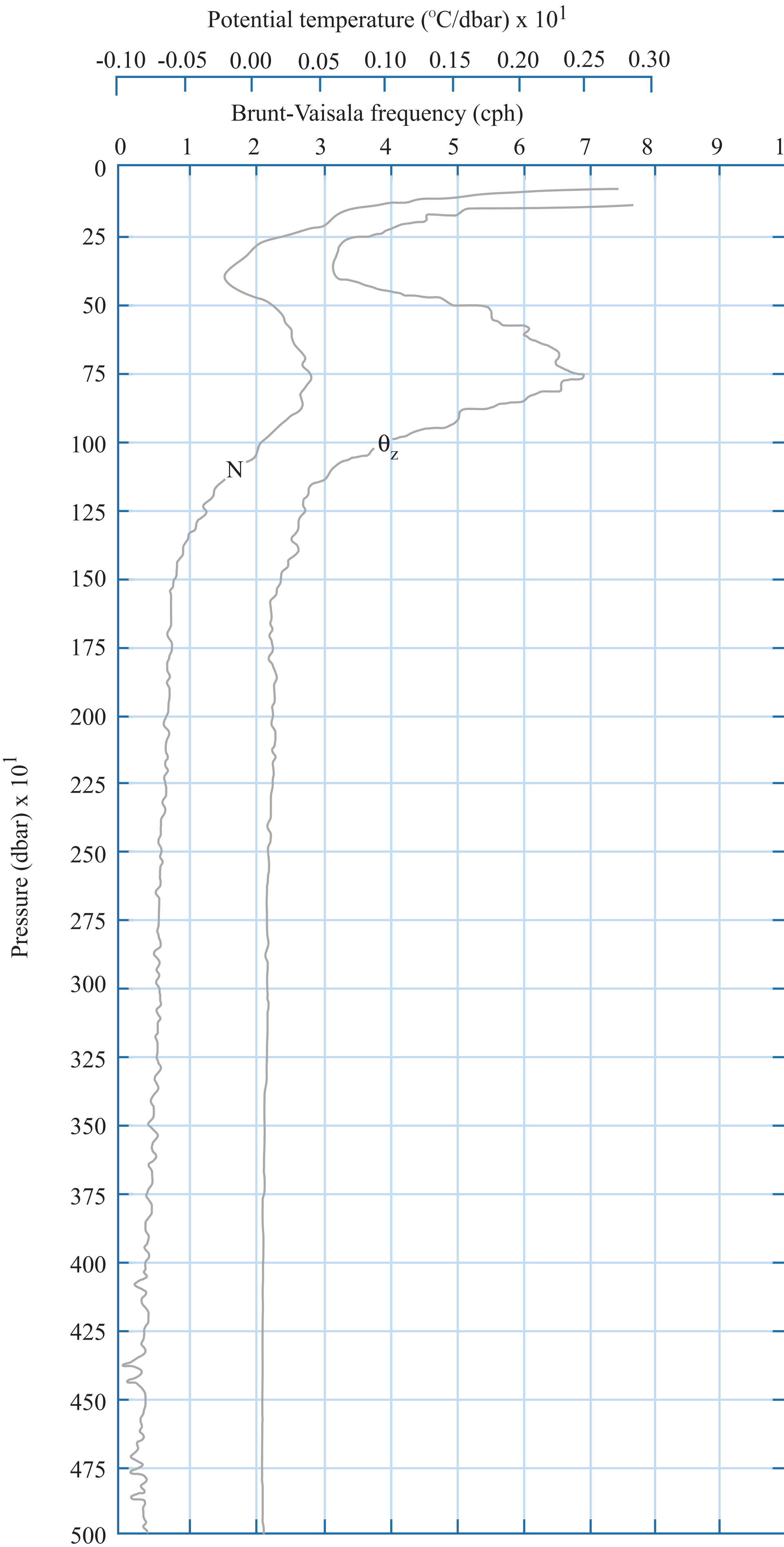
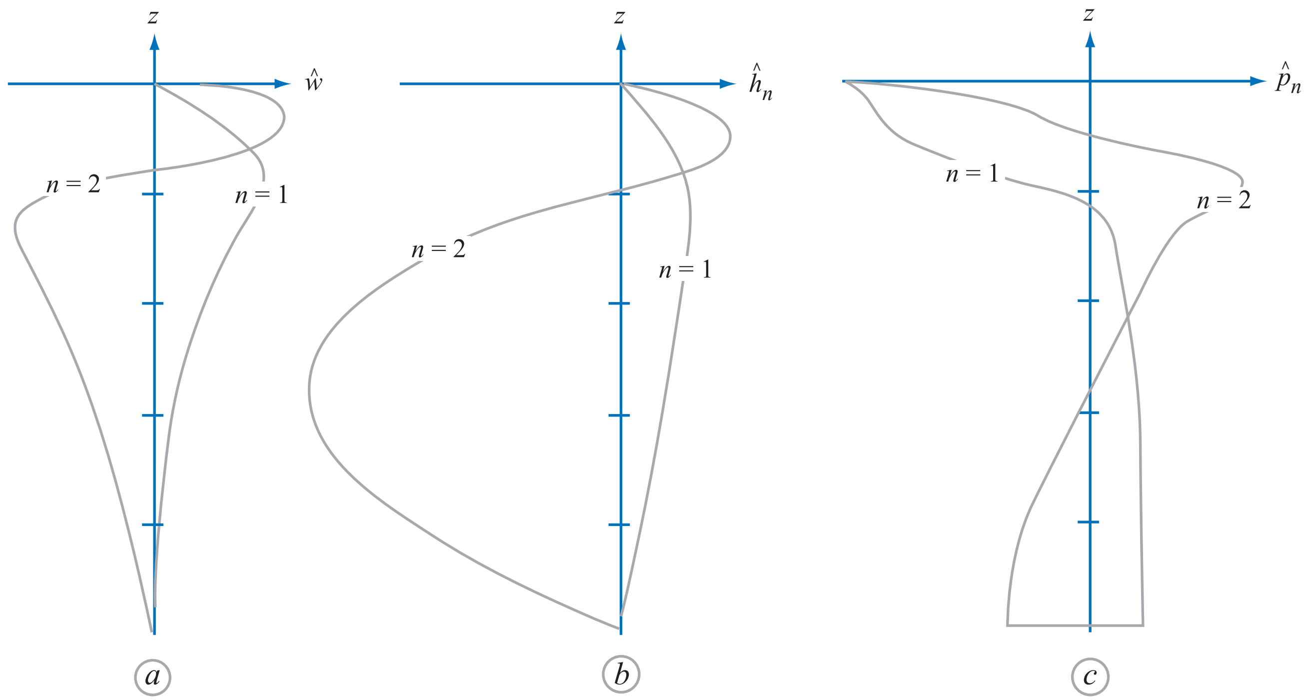


The distribution of N(z) in North Atlantic

Figure by MIT OpenCourseWare.



Eigenfunctions for the buoyancy profile of figure 9.10

The figures on the left of the figure are essentially W_j. The second figure is essentially the form of the solution in the long wave limit, and the last figure is the shape of the pressure or horizontal velocity in each mode, really the derivative of the function W. Note, as expected, the n = 1 mode has no zeros for W (just like $\sin \pi z / D$), while the second mode has a single zero (like $\sin 2\pi z / D$). However, the location of the zero and maxima differ from the constant N case. Note too that at great depth where N is small, the eigenfunctions are not oscillatory in agreement with out qualitative discussion of the governing equation.

Figure by MIT OpenCourseWare.