

Lecture 7

7.1 Administration

- Hand back Q3, PS3.
- No class next Tuesday (October 7th).
- Class a week from Thursday (October 9th) will be a guest lecturer.
- Last question in PS4: Only take body force to $\nabla \cdot \tau$ stage.

7.2 Continuity equation

Ended last class with the continuity equation:

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0 \quad (7.1)$$

or

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \quad (7.2)$$

7.3 Boussinesq approximation

I want to take an aside and say a few words about the Boussinesq approximation. Imagine a 3D fluid, like the ocean, where density can be broken into three terms:

$$\rho = \rho_o + \tilde{\rho}(z) + \rho'(x, y, z, t) \quad (7.3)$$

where

$$\rho_o \ll \tilde{\rho} \ll \rho' \quad (7.4)$$

This requires that the vertical variation of density is small relative to the mean, and that the horizontal and temporal variations are small relative to the vertical. This is a good approximation for the ocean where we have

$$\rho_o \cong 1000 \text{ kg/m}^3 \quad (7.5)$$

$$\tilde{\rho} \cong 10 \text{ kg/m}^3 \quad (7.6)$$

$$\rho' \cong 0.1 \text{ kg/m}^3 \quad (7.7)$$

But not so good for the atmosphere over the bottom 15 km where

$$\rho_o \cong 0.5 \text{ kg/m}^3 \quad (7.8)$$

$$\tilde{\rho} \cong 0.5 \text{ kg/m}^3 \quad (7.9)$$

$$\rho' \cong 0.005 \text{ kg/m}^3 \quad (7.10)$$

Lets start with the ocean (we'll come back to the atmosphere). For simplicity, subsume $\bar{\rho}$ into ρ' such that

$$\rho = \rho_o + \rho'(x, y, z, t) \quad (7.11)$$

where

$$\rho_o \ll \rho' \quad (7.12)$$

Note this last statement (equation 7.12) is part of the Boussinesq approximation.

For the impact of this statement, we have to consider each relevant equation. For the continuity equation we get

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \quad (7.13)$$

$$\frac{\partial}{\partial t}(\rho_o + \rho') + \nabla \cdot ((\rho_o + \rho')\mathbf{u}) = 0 \quad (7.14)$$

$$\frac{\partial \rho'}{\partial t} + \nabla \cdot (\rho_o \mathbf{u}) + \nabla \cdot (\rho' \mathbf{u}) = 0 \quad (7.15)$$

Here the $\frac{\partial \rho_o}{\partial t}$ term is dropped since ρ_o is constant. Scaling tells us that

$$\frac{\partial \rho}{\partial t} \rightarrow \frac{\rho}{T} \rightarrow \frac{\rho}{L/U} \rightarrow \rho \frac{U}{L} \quad (7.16)$$

$$\nabla \cdot (\rho \mathbf{u}) \rightarrow \rho \frac{U}{L} \quad (7.17)$$

Hence, the continuity equation scales as

$$\frac{\partial \rho'}{\partial t} + \nabla \cdot (\rho_o \mathbf{u}) + \nabla \cdot (\rho' \mathbf{u}) = 0 \quad (7.18)$$

$$\rho' \frac{U}{L} \quad \rho_o \frac{U}{L} \quad \rho' \frac{U}{L} \quad (7.19)$$

But $\rho' \ll \rho_o$ and ρ_o is constant, so we are just left with

$$\nabla \cdot (\rho_o \mathbf{u}) = 0 \quad (7.20)$$

$$\nabla \cdot \mathbf{u} = 0 \quad (7.21)$$

According to this Boussinesq approximation, the continuity equation reduces to an expression that the flow is incompressible. That is, fluid chunks change their shape, but not their volume. Thus, here, conservation of mass becomes conservation of volume. In 2D

$$\nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (7.22)$$

and figure 7.1 shows some of the implications pictorially.

7.3.1 Sound waves

As an aside, for the equation of state of sea water

$$\rho = \rho(p, T, S) \quad (7.23)$$

we have

$$\frac{\partial \rho}{\partial t} = \frac{\partial \rho}{\partial p} \Big|_{T,S} \frac{\partial p}{\partial t} + \frac{\partial \rho}{\partial T} \Big|_{p,S} \frac{\partial T}{\partial t} + \frac{\partial \rho}{\partial S} \Big|_{p,T} \frac{\partial S}{\partial t} \quad (7.24)$$

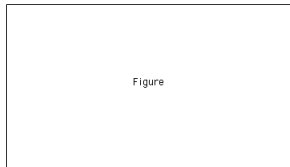


Figure 7.1: (**fig:Lec7BoussinesqContinuity**) Two implications of Boussinesq approximation on the continuity equation in 2D. (Above) If there is a compressive force along only one axis, the fluid will stretch along the orthogonal axis. (Below) If there is equal compressive force along each axis, then $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0$.

(**Jim - I didn't know how to sink the variables that are held constant to the bottom of the vertical bar as desired, and after asking around, and a little googling, still had no luck – maybe the folks at OpenCourse know how to do it**) which implies $\frac{\partial p}{\partial t} = 0$. But this means no sound waves! Note that is also implies $\frac{\partial T}{\partial t} = 0$, which is not true. $\rho_o \ll \rho(x, y, z, t)$ is only part of Boussinesq. We will also have to bring in the hydrostatic balance, but much more on that later.

7.3.2 Atmosphere: the anelastic approximation

Boussinesq is okay for the ocean, but not so good for the atmosphere. In particular, the difference between ρ_o and $\tilde{\rho}(z)$ are large. Instead, assume

$$\rho = \bar{\rho}(z) + \rho'(x, y, z, t), \quad \bar{\rho} \gg \rho' \quad (7.25)$$

Substitution yields (recalling that $\bar{\rho}$ does not depend on t):

$$\frac{\partial \rho'}{\partial t} + \nabla \cdot (\bar{\rho} \mathbf{u}) + \nabla \cdot (\rho' \mathbf{u}) = 0 \quad (7.26)$$

Scale analysis shows us that

$$\frac{\partial \rho'}{\partial t} \rightarrow \rho' \frac{U}{L}, \quad \nabla \cdot (\bar{\rho} \mathbf{u}) \rightarrow \bar{\rho} \frac{U}{L}, \quad \nabla \cdot (\rho' \mathbf{u}) \rightarrow \rho' \frac{U}{L} \quad (7.27)$$

Since $\bar{\rho} \gg \rho'$, we have $\nabla \cdot (\bar{\rho} \mathbf{u}) = 0$, or

$$\bar{\rho} \nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla \bar{\rho} = 0 \quad (7.28)$$

This is called the anelastic approximation; volume is not conserved, but conservation of vertical part of mass occurs. (**Jim – Does this make sense?**) For a diagram of conservation of mass under different approximations, see figure 7.2.

7.4 Back to the Reynold's transport theorem

The RTT is not limited to its application to mass conservation, but can also be used for balance problems. RTT says

$$\frac{dC}{dt} = \iiint_V \left[\frac{Dc}{Dt} + c \nabla \cdot \mathbf{u} \right] dV \quad (7.29)$$

If there is conservation of C , then $\frac{dC}{dt} = 0$. If there is a balance, then $\frac{dC}{dt} =$ sources and sinks, commonly denoted by Q . Consider

$$\frac{dC}{dt} = Q \quad (7.30)$$

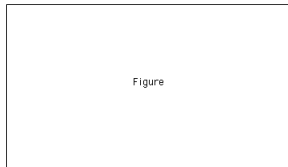


Figure 7.2: (`fig:Lec7MassConsrvdDiffApprox`) Conservation of mass under different approximations. Topmost is no approximation, where mass is conserved, but not volume. In the middle, there is the Boussinesq approximation, where volume is conserved, but not mass (**Jim – What do you mean by mass is not conserved; I though that this was all derived from the statement of conservation of mass – the continuity equation**). At the bottom there is the analastic approximation. This can be thought of as not conserving mass or volume *or* conserving some volume and some mass.

Using the RTT

$$\iiint_V \left[\frac{Dc}{Dt} + c \nabla \cdot \mathbf{u} \right] dV = \iiint_V q dV \quad (7.31)$$

where q is the per unit volume version of Q . This must hold for all volumes. Thus

$$\frac{Dc}{Dt} + c \nabla \cdot \mathbf{u} = q \quad \text{or} \quad \frac{\partial c}{\partial t} + \nabla \cdot (c \mathbf{u}) = q \quad (7.32)$$

For linear momentum (recall from 2nd class, can also apply RTT on energy and entropy)

$$\frac{d\mathbf{P}}{dt} |_{\mathcal{V}} = \mathbf{F} \quad (7.33)$$

$$= \mathbf{F}_B + \mathbf{F}_S + \mathbf{F}_L \quad (7.34)$$

$\mathbf{F}_B \equiv$ body force. That is, action from a distance. Examples: gravity, magnetism, etc.

$\mathbf{F}_S \equiv$ surface force. That is, forces acting on a fluid element from direct contact.

$\mathbf{F}_L \equiv$ line force. Forces acting on a line. Surface tension is the only one I can think of, and I don't plan on talking about it in this course.

So

$$\frac{d\mathbf{P}}{dt} |_{\mathcal{V}} = \mathbf{F}_B + \mathbf{F}_S \quad \text{or} \quad \frac{d(M\mathbf{u})}{dt} |_{\mathcal{V}} = \mathbf{F}_B + \mathbf{F}_S \quad (7.35)$$

Now, lets use the RTT! The extensive property is $M\mathbf{u}$, while the intensive property is $\rho\mathbf{u}$. Thus

$$\frac{d(M\mathbf{u})}{dt} = \iiint_V \left[\frac{\partial(\rho\mathbf{u})}{\partial t} + \nabla \cdot (\rho\mathbf{u}\mathbf{u}) \right] dV = \mathbf{F}_B + \mathbf{F}_S \quad (7.36)$$

Using a component by component consideration of the vector identity of

$$\nabla \cdot (a\mathbf{b}) = (\mathbf{b} \cdot \nabla)a + a(\nabla \cdot \mathbf{b}) \quad (7.37)$$

we get

$$\nabla \cdot (a\mathbf{b}) = \mathbf{b}(\nabla \cdot a) + (a \cdot \nabla)\mathbf{b} \quad (7.38)$$

and hence

$$\frac{d(M\mathbf{u})}{dt} = \iiint_V \left[\rho \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \frac{\partial \rho}{\partial t} + \mathbf{u}(\nabla \cdot \rho \mathbf{u}) + (\rho \mathbf{u} \cdot \nabla) \mathbf{u} \right] dV = \mathbf{F}_B + \mathbf{F}_S \quad (7.39)$$

$$\Rightarrow \frac{d(M\mathbf{u})}{dt} = \iiint_V \left[\rho \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \left(\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{u} \right) + \rho(\mathbf{u} \cdot \nabla) \mathbf{u} \right] dV = \mathbf{F}_B + \mathbf{F}_S \quad (7.40)$$

$$(7.41)$$

Recall from continuity that

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{u} = 0 \quad (7.42)$$

Hence

$$\frac{d(M\mathbf{u})}{dt} = \iiint_V \left[\rho \frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u} \cdot \nabla) \mathbf{u} \right] dV = \mathbf{F}_B + \mathbf{F}_S \quad (7.43)$$

$$(7.44)$$

On the LHS we have the local change of momentum and the change of momentum due to the advection of velocity. Factoring out ρ yields

$$\iiint_V \rho \frac{D\mathbf{u}}{Dt} dV = \mathbf{F}_B + \mathbf{F}_S \quad (7.45)$$

$$(7.46)$$

Thus, we are left with the material derivative for velocity (**REALLY STRESS THIS**). This happens whenever we are applying the RTT to an intensive property that is a product of density, as is seen in the following aside.

Aside: Consider an extensive property, F , where the intensive equivalent is ρf . Applying the RTT we get

$$\frac{DF}{Dt} = \iiint_V \left[\frac{\partial(\rho f)}{\partial t} + \nabla \cdot (\rho f \mathbf{u}) \right] dV \quad (7.47)$$

Recalling the vector identity $\nabla \cdot (a\mathbf{b}) = (\mathbf{b} \cdot \nabla)a + a(\nabla \cdot \mathbf{b})$ from before, we get

$$\frac{DF}{Dt} = \iiint_V \left[\rho \frac{\partial f}{\partial t} + f \frac{\partial \rho}{\partial t} + (\rho \mathbf{u} \cdot \nabla)f + f(\nabla \cdot \rho \mathbf{u}) \right] dV \quad (7.48)$$

$$\frac{DF}{Dt} = \iiint_V \left[\rho \frac{\partial f}{\partial t} + f \left(\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{u} \right) + (\rho \mathbf{u} \cdot \nabla)f \right] dV \quad (7.49)$$

Again, apply continuity ($\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{u} = 0$)

$$\frac{DF}{Dt} = \iiint_V \left[\rho \frac{\partial f}{\partial t} + (\rho \mathbf{u} \cdot \nabla)f \right] dV \quad (7.50)$$

$$\frac{DF}{Dt} = \iiint_V \rho \frac{Df}{Dt} dV \quad (7.51)$$

$$(7.52)$$

This is yet another form of the RTT, which we arrive at *only* when the intensive variable is a product of ρ . End aside.

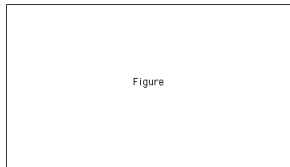


Figure 7.3: (`fig:Lec7SurfaceForces`) Surface stresses acting on a fluid. Normal or compressive stress ($\sigma \equiv \tau_{ii}$) on the left and shear stress on the right ($\sigma \equiv \tau_{ij}$).

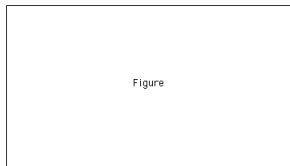


Figure 7.4: (`fig:Lec7Stresses`) Usual picture of stresses on a fluid blob. Here normal and shear stresses are written as τ_{ij} where normal stress are those where $i = j$, and shear stresses those when $i \neq j$. $i, j = 1, 2, 3 = x, y, z$ -directions, respectively. Thus, τ_{22} is the normal stress in the y -direction, and τ_{13} is the shear stress on the x -plane (x normal to that particular plane) in the z -direction.

Back to momentum:

$$\iiint_V \rho \frac{D\mathbf{u}}{Dt} dV = \mathbf{F}_B + \mathbf{F}_S \quad (7.53)$$

First consider body forces. In fluids that we are interesting in, the only “force at a distance” is gravity. Gravity acts on all the mass in the blob, so

$$\mathbf{F}_B = \iiint_V \rho \mathbf{g} dV \quad \text{where} \quad \mathbf{g} = 0\mathbf{i} + 0\mathbf{j} - g\mathbf{k} \quad (7.54)$$

That one was easy. \mathbf{F}_S is a little trickier. Its the force acting on the surface of the blob, so we know we’re going to have an area integral

$$\mathbf{F}_S = \iint_A d\mathbf{A} \quad (7.55)$$

What goes inside the integral? You might recall back from the 2nd lecture of the class the concept of a normal stress and a shear stress (force/area), as is seen in figure 7.3. The usual picture that is drawn is that as seen in figure 7.4. With this, we can define a *symmetric* stress tensor

$$\boldsymbol{\tau} = \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix} \neq \mathbf{e} \quad (7.56)$$

Note that this is stress, *not* strain. The stress, τ_{ij} , or $\boldsymbol{\tau}$ is the surface force per unit area that goes into our integral

$$\mathbf{F}_S = \iint_A \boldsymbol{\tau} d\mathbf{A} \quad (7.57)$$

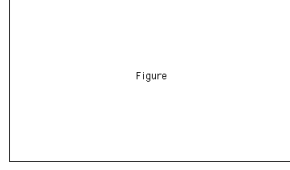


Figure 7.5: (**fig:Lec7Stress2D**) Diagram of stress in two dimensions. Same as figure 7.4, but in two dimensions and $i, j = 1, 2$ written explicitly as $i, j = x, y$.

Aside: An two dimensions we have the case shown in figure 7.5, and $\tau \cdot d\mathbf{A}$ becomes

$$\begin{aligned} \tau \cdot d\mathbf{A} &= \begin{bmatrix} \tau_{xx} & \tau_{xy} \\ \tau_{yx} & \tau_{yy} \end{bmatrix} \cdot \begin{bmatrix} dA_x \\ dA_y \end{bmatrix} \\ &= \sum_{i=1}^2 \delta_i \left(\sum_{j=1}^2 \tau_{ij} dA_j \right) \quad (\text{by definition}) \\ &= (\tau_{xx} dA_x + \tau_{xy} dA_y) \mathbf{i} + (\tau_{yx} dA_x + \tau_{yy} dA_y) \mathbf{j} \\ &= [\tau_{xx} \ \tau_{xy}] \cdot d\mathbf{A} \mathbf{i} + [\tau_{yx} \ \tau_{yy}] \cdot d\mathbf{A} \mathbf{j} \end{aligned} \tag{7.58}$$

$$\tag{7.59}$$

End aside.

Thus, the RHS of the momentum equation (7.53) becomes

$$\text{RHS} = \iiint_V \rho \mathbf{g} dV + \iint_A \tau d\mathbf{A} \tag{7.60}$$

Using the divergence theorem

$$\iint_A \tau d\mathbf{A} = \iiint_V \nabla \cdot \tau dV \tag{7.61}$$

But is this really OK for tensors? Yes:

$$\begin{aligned} \nabla \cdot \tau &= \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{bmatrix} \begin{bmatrix} \tau_{xx} & \tau_{xy} \\ \tau_{yx} & \tau_{yy} \end{bmatrix} \\ &= \sum_{i=1}^2 \delta_i \left(\sum_{j=1}^2 \frac{\partial}{\partial x_j} \tau_{ij} \right) \\ &= \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} \right) \mathbf{i} + \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} \right) \mathbf{j} \\ &= \nabla_{xy} \cdot [\tau_{xx} \ \tau_{yx}] \mathbf{i} + \nabla_{xy} \cdot [\tau_{xy} \ \tau_{yy}] \mathbf{j} \end{aligned} \tag{7.62}$$

$$\tag{7.63}$$

Since $\tau_{yx} = \tau_{xy}$, this is consistent with $\iint \tau \cdot d\mathbf{A} = \iiint \nabla \cdot \tau dV$. So, the RHS of the momentum equation (7.53) becomes

$$\begin{aligned} \text{RHS} &= \iiint_V \rho \mathbf{g} dV + \iiint_V \nabla \cdot \tau dV \\ \Rightarrow \iiint_V \rho \frac{D\mathbf{u}}{Dt} dV &= \iiint_V [\rho \mathbf{g} + \nabla \cdot \tau] dV \end{aligned} \tag{7.64}$$

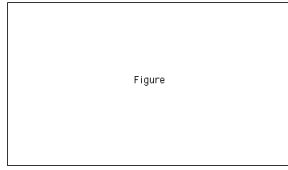


Figure 7.6: (fig:Lec7StressStrain) Linear relationship between stress and strain.

This expression must hold for all volumes, so

$$\rho \frac{D\mathbf{u}}{Dt} = \rho \mathbf{g} + \nabla \cdot \boldsymbol{\tau} \quad (7.65)$$

We're getting there! All that is left is to come up with an expression for $\boldsymbol{\tau}$, the stress tensor. Remember how much time we spent showing how velocity changes are related to rates of strain? Velocity gradients and all that? Now we're going to spend time showing how stress is related to rates of strain. When we're done we'll have a relationship between velocity changes and stress... just what we want!

The pictures that we drew earlier show us that there is a linear relationship between stress and strain (see figure 7.6). The so-called "constitutive equation" (the relationship between stress and strain) for fluids is only slightly more complex than Hooke's Law.

7.5 Constitutive relationship for fluids

$$\boldsymbol{\tau} = a \mathbf{e} + b \mathbf{I} \quad (7.66)$$

where $\mathbf{e} \equiv$ strain rate tensor
 $\mathbf{I} \equiv$ identity matrix
 $a, b \equiv$ constants

This is the simplest linear equation relating stress and strain that includes a term that allows a stress in the absence of motion ($\mathbf{e} = 0$). It assumes that fluid is isotropic. That is, it behaves in the same way in all directions. This might not be a good assumption for magnetized or charged fluids in an external field, for example. For the atmosphere and ocean, it works just fine. From KC01, page 95:

"We shall assume a linear relationship of the type $\sigma_{ij} = K_{ijmn} e_{mn}$, where K_{ijmn} is a 4th order tensor having 81 unknowns..."

"It will now be shown that only two of the 81 elements of K_{ijmn} survive if it is assumed that the medium is isotropic and the stress tensor is symmetric."

Since $\boldsymbol{\tau}$ is 3×3 , this equation (7.66) actually represents 9 equations:

$$\tau_{xx} = a e_{xx} + b \quad (7.67)$$

$$\tau_{xy} = a e_{xy} + 0 \quad (7.68)$$

$$\tau_{xz} = a e_{xz} + 0 \quad (7.69)$$

$$\tau_{yx} = a e_{yx} + 0 \quad (7.70)$$

$$\tau_{yy} = a e_{yy} + b \quad (7.71)$$

\vdots

$$\tau_{zz} = a e_{zz} + b \quad (7.72)$$

Add together equations 7.67, 7.71, and 7.72:

$$\begin{aligned}
 \tau_{xx} + \tau_{yy} + \tau_{zz} &= a(e_{xx} + e_{yy} + e_{zz}) + 3b \\
 &= a\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + 3b \\
 &= a\nabla \cdot \mathbf{u} + 3b
 \end{aligned} \tag{7.73}$$

Stokes (bless his soul) proved that $a = 2\mu$, where $\mu \equiv$ viscosity of lecture 3. (Part of Stokes' proof boiled those 81 unknowns down to 2). (**Jim – A reference here to his proof might be nice**). Now we can solve for b :

$$\begin{aligned}
 \tau_{xx} + \tau_{yy} + \tau_{zz} &= 2\mu\nabla \cdot \mathbf{u} + 3b \\
 \Rightarrow b &= \frac{1}{3}(\tau_{xx} + \tau_{yy} + \tau_{zz}) - \frac{2}{3}\mu\nabla \cdot \mathbf{u}
 \end{aligned} \tag{7.74}$$

By definition, pressure at a point in a fluid is the negative of the mean of the three normal stresses:

$$\Rightarrow p = -\frac{1}{3}(\tau_{xx} + \tau_{yy} + \tau_{zz}) \tag{7.75}$$

$$\Rightarrow b = -p - \frac{2}{3}\mu\nabla \cdot \mathbf{u} \tag{7.76}$$

Thus the constitutive relationship becomes

$$\tau = 2\mu\mathbf{e} - \left(p + \frac{2}{3}\mu\nabla \cdot \mathbf{u}\right)\mathbf{I} \tag{7.77}$$

7.6 Reading for class 8

KC01: 4.13 - 4.15