

Lecture 3

3.1 Administration

- Role call. Try to get names right.
- **Homework Hints:** Problem 3.9 is bizarre. In order to get the right answer, I had to go to a very non-standard geometry (see figure 3.1 and equations 3.1-3.3). Note: Not even right handed!

$$x = r \cos \theta \tag{3.1}$$

$$y = r \sin \theta \sin \phi \tag{3.2}$$

$$z = r \sin \theta \cos \phi \tag{3.3}$$

Thus corresponding area element ($d\mathbf{A}$) is given by:

$$\begin{aligned} d\mathbf{A} &= dA \mathbf{n} \\ &= r dA (\cos \theta \mathbf{i} + \sin \theta \sin \phi \mathbf{j} + \sin \theta \cos \phi \mathbf{k}) \end{aligned} \tag{3.4}$$

$$\text{where } dA = (r d\theta)(r \sin \theta d\phi) \tag{3.5}$$

- Correction from last time: (**Lecture 2 notes are updated to reflect this!**)

$$\begin{aligned} a &= \text{Re}[A_c e^{i(kx+ly-\omega t)}] \\ A_c &= A_1 - iA_2 = A e^{i\phi} \end{aligned} \tag{3.6}$$

- Quiz 1!

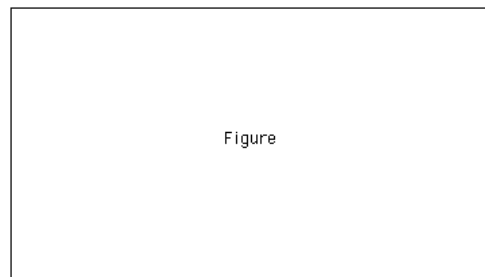
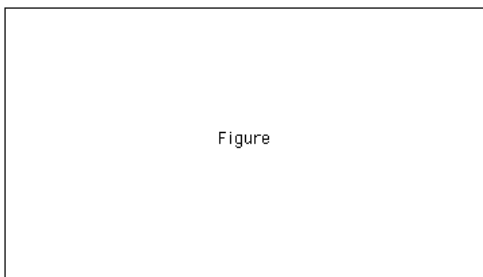


Figure 3.1: (`fig:NonStandardSphericalCoordinates`, `fig:NonStandardAreaElement`) Non-standard spherical coordinate system (left panel) and diagram for a corresponding area element (right panel).

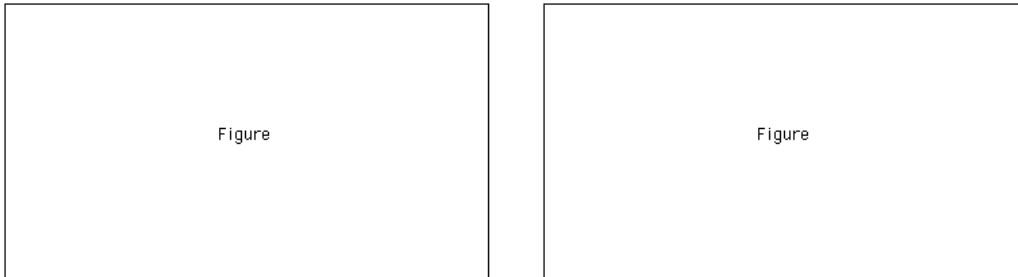


Figure 3.2: (fig:ReferenceFrame1, fig:ReferenceFrame2) Stationary reference frame (left panel) and moving reference frame (right panel). The stationary reference (xyz) frame could be a guy on the side of a road, the moving frame ($x'y'z'$) a truck, and P a baseball.

3.2 Scale analysis

A close cousin of dimensional analysis is scale analysis. In dimensional analysis we come up with important dimensionless groups that are applicable to a particular problem. In scale analysis, we have an equation, or set of equations, that completely describe a system, and we want to see what terms we can neglect based on their size.

3.2.1 A simple example

In the inertial frame, xyz , the speed of a particle P is measured as u . What is the speed of P in the $x'y'z'$ reference frame, moving at a speed v with respect to xyz ? (See figure 3.2.)

Special relativity tells us the particle velocity in the moving frame u' is given by

$$u' = \frac{u - v}{1 - \frac{uv}{c^2}} \quad (3.7)$$

There are many ways to approach this problem. The easiest is to make the following scale argument:

- U is a typical velocity seen in the xyz reference frame, and is of the order of the relative velocity of the frames. Thus, u , v , and u' are all of the order U , sometimes denoted $O(U)$.
- The speed of light is given by c , where $c \gg U$.

Thus equation 3.7 scales in the following manner:

$$U = \frac{U - U}{1 - \frac{U^2}{c^2}} \quad (3.8)$$

Since $c \gg U$, then $U/c \ll 1$ and thus the “correction term” is negligible compared to 1 and the relative velocity equation reduces to the expected result of

$$u' = u - v \quad (3.9)$$

Yes, we could have done this by inspection, but I wanted to give an easy example. Note that of course equation 3.9 breaks down when we do not have the case where $c \gg U$.

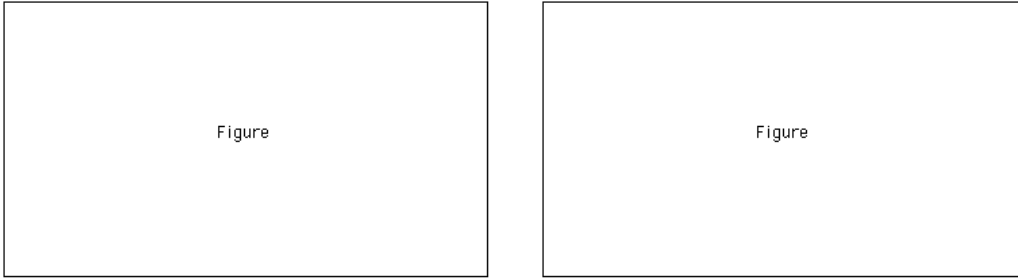


Figure 3.3: (fig:DistanceScales1, fig:DistanceScales2) Horizontal and vertical scales of a geophysical fluid.

3.2.2 A geophysical example

The fluid momentum equation in the x directions is

$$\frac{Du}{Dt} = -\frac{1}{\rho_o} \frac{\partial p}{\partial x} - 2\Omega w \cos \phi + 2\Omega v \sin \phi + \nu \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right] \quad (3.10)$$

With characteristic scales of U and L for horizontal velocity and distance scales and W and H the vertical velocity and distance scales, the scaled terms of momentum equation (equation 3.10) are those shown in equation 3.11. For now, you will have to take my word that this is indeed how the equation is scaled.

$$\frac{U}{L} = \frac{p}{\rho_o L} \quad \Omega W \quad \Omega U \quad \nu \frac{U}{L^2} \quad \nu \frac{U}{L^2} \quad \nu \frac{U}{H^2} \quad (3.11)$$

Now we look to see how the different scales compare with each other to see which terms we might be able to drop:

- In geophysical flow, $L \gg H$ (see figure 3.3).
- If you assume an incompressible fluid, it follows that $U \gg W$. For now you must take my word for this.

With this information, the following simplifications can be made. Since $U \gg W$, we can ignore the ΩW term since it will be small relative to ΩU . Since $L \gg H$, we can ignore the $\nu U/L^2$ terms as they will be negligible relative to the $\nu U/H^2$ term. Thus, momentum equation in the x direction (3.10) is simplified to the following:

$$\frac{Du}{Dt} = -\frac{1}{\rho_o} \frac{\partial p}{\partial x} + 2\Omega v \sin \phi + \nu \frac{\partial^2 u}{\partial z^2} \quad (3.12)$$

Note that you can sometimes also get rid of the last term if you can argue that the vertical shear is small.

3.3 What is a fluid?

Enough of the review stuff. Let's talk about fluids. A sensible place to start is to ask the question, "what is a fluid?". We can define all (macroscopic) material as either solid, liquid, or gas.

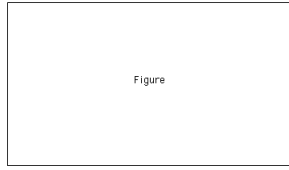


Figure 3.4: (`fig:ReactionToCompressiveForce`) Fluids and solids act the same in response to a compressive force. Here Δx is the deformation of the material and σ is the compressive force per area (aka the normal force).

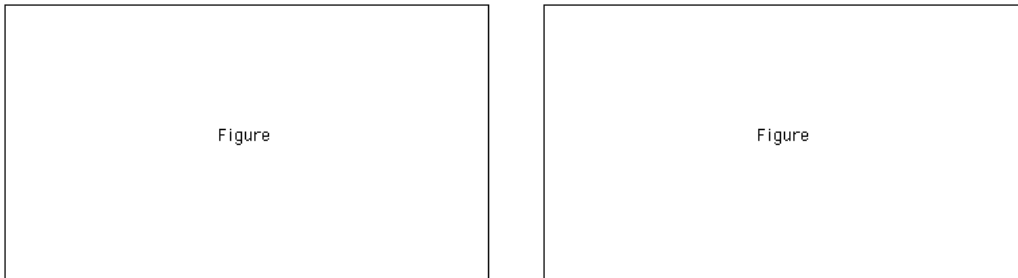


Figure 3.5: (`fig:NoShearOnSolid`, `fig:ShearOnSolid`) Shear on a solid. τ_{31} is the shear stress operating in the z plane (3) in the x direction (1). Note that in figure 3.4, $\sigma = \tau_{33}$.

- Liquids and gases are both considered fluids... They have similar dynamic behavior.
- The key difference between a solid and a fluid is the mode of resistance to change of shape.

We'll assume a continuum hypothesis: Fluids are made up of interacting molecules, and a continuum exists if the individual effect of a molecule on density, pressure, temperature, etc. are negligible. Just like in thermodynamics, we are looking at bulk measures of the fluid rather than explicitly accounting for the behavior of each molecule.

Back to mode of resistance. Under a compressive force, solids and fluids act the same (see figure 3.4). Stress and deformation are related by Hooke's Law:

$$\sigma = E\Delta x \tag{3.13}$$

Here E is the modulus of elasticity and is material dependent, and for the most part, empirically determined. So solids and fluids behave the same under a normal stress, but what about a shear stress?

First consider a solid. Imagine a block affixed to the table where I apply a constant shear stress to the top surface. What happens? The block might deform a bit, but eventually it will stop moving. The shear stress applied to the top of the block will be balanced by the block's rigidity (see figure 3.5).

As seen in figure 3.5, τ_{31} is the shear stress operating in the z plane (3) in the x direction (1). With γ the displacement angle (in radians) and G the shear modulus of elasticity, Hooke's Law for shear is given as follows:

$$\tau_{31} = G\gamma \tag{3.14}$$

Now let's look at a fluid. A solid has a rigidity that can balance the applied shear stress. A fluid does not. There is no proportionality between shear stress and the deformation angle because

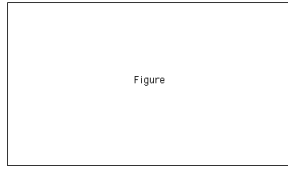


Figure 3.6: (fig:FluidDeformation) Rate of change of the deformation angle of a fluid.

there is no deformation angle. The proportionality is instead between the shear stress and the rate of change of the deformation angle (see figure 3.6):

$$\tau_{31} = \mu \dot{\gamma} \quad (3.15)$$

With the geometry seen in figure 3.6, the fluid shear stress equation 3.15 becomes:

$$\begin{aligned} \Delta x &= \frac{\Delta u}{\Delta z} \Delta z \Delta t \\ \Delta \gamma &\cong \frac{\Delta x}{\Delta z} = \frac{\frac{\Delta u}{\Delta z} \Delta z \Delta t}{\Delta z} = \frac{\Delta u \Delta t}{\Delta z} \\ \Rightarrow \frac{\Delta \gamma}{\Delta t} &= \frac{\frac{\Delta u}{\Delta z} \Delta t}{\Delta t} = \frac{\Delta u}{\Delta z} \\ \Rightarrow \frac{d\gamma}{dt} &= \frac{du}{dz} \\ \Rightarrow \tau_{31} &= \mu \dot{\gamma} = \mu \frac{du}{dz} \end{aligned} \quad (3.16)$$

This is the mathematical definition of a fluid. When a shear stress is applied to a fluid, the fluid deforms continuously. The viscosity, μ , is a measure of the resistance of the fluid to flow. Noting that $\tan \gamma \cong \gamma = \Delta x / \Delta z = dx / dz$ in the limit of infinitesimally small Δz (**Does that note make sense Jim?**), in summary we have:

Stress	Solid	Fluid
Compressive	$\tau_{31} = G\gamma$	$\tau_{31} = \mu \dot{\gamma}$
Shear	$\tau_{31} = G \frac{dx}{dz}$	$\tau_{31} = \mu \frac{du}{dz}$

Table 3.1: Summary of reaction of solids and fluids to compressive and shear stress.

Note that in the solid, deformation is due to displacement gradients, and in the fluid they are due to velocity gradients. In solid mechanics, which you are all familiar with, displacements are the dependent variables (functions of time). In fluid mechanics, velocities are the dependent variables (functions of locations and time). These differences can be characterized as Lagrangian (solids) and Eulerian (fluids), and we'll talk about them in depth in a bit. Fluids can be characterized by the relationship between shear stress and rate of deformation (see figure 3.7).

3.4 Classical laws of physics

Fluids are subject to the same laws of physics as solids, but the difference between solids and fluids discussed section 3.3 means that the analytic forms of the laws can look quite different. Actually, one can think of fluids in a solids-like framework. Instead of considering fluids passing through a volume, consider a marked chunk of fluid following the fluid flow (figure 3.8). This allows us to write down various conservation relationships in familiar forms:

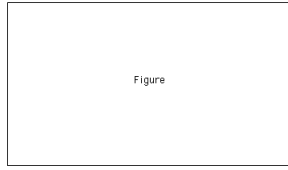


Figure 3.7: (fig:FluidTypes) Fluid types.

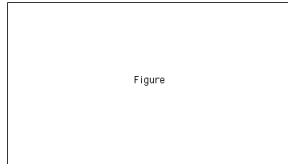


Figure 3.8: (fig:FollowFluidParcel) Following a fluid parcel.

Mass:	$\frac{dM}{dt} _v = 0$	$M \equiv \text{mass}$
Linear momentum: (motion stays in motion)	$\bar{F} = \frac{d\bar{P}}{dt} _v$	$\bar{F} \equiv \text{force}$ $\bar{P} \equiv \text{momentum}$
Angular momentum: (useful for our purposes)	$\bar{T} = \frac{d\bar{H}}{dt} _v$	$\bar{T} \equiv \text{torque on system}$ $\bar{H} \equiv \text{angular momentum}$ $E \equiv \text{total energy}$
1 st Law (energy):	$\dot{Q} - \dot{W} = \frac{dE}{dt} _v$	$\dot{Q} \equiv \text{rate of heat transfer (+ when added)}$ $\dot{W} \equiv \text{rate of work (+ when added by system)}$
2 nd Law (energy):	$\frac{dS}{dt} _v \geq \frac{\dot{Q}}{T}$	$\dot{S} \equiv \text{entropy of system}$

These equations should look familiar, and we'll return to many of them in the next few weeks, transforming these familiar equations into ones more suitable for our purposes.

3.5 Challenge in studying fluid mechanics

Given that we now know what a fluid is, what makes the study of fluids dynamics difficult and challenging? In a nutshell, the problem is nonlinear advection (**Jim: Up until this point you have not defined advection**). Fluid flow can have incredibly complex structure. The ultimate manifestation of this is turbulence (see figures 3.9-3.12). (**Jim: You have a small note here saying “less of a prob w/Lagrangian”. Do you want this mentioned/elaborated without having yet defined Lagrangian?**).

According to the Institute of Physics, the ten great unsolved problems in physics are (we have 30% of them cover here in EAPS!): (**Jim: You have 4, 5, 8, and 9 arrowed in your notes... should it be 40%?**).

1. Quantum gravity
2. Understanding the nucleus
3. Fusion energy

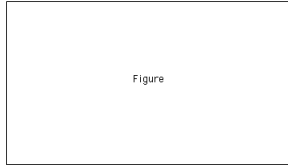


Figure 3.9: (`fig:TurbulenceGeneral`) Example of turbulence.

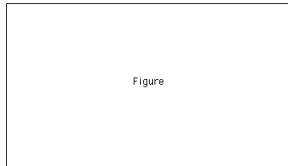


Figure 3.10: (`fig:TurbulenceJet`) Turbulence from a jet.

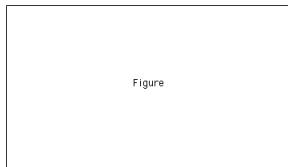


Figure 3.11: (`fig:TurbulenceOcean`) Non-linearity in the ocean.

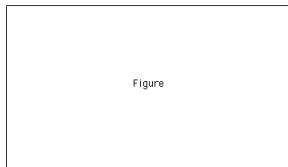


Figure 3.12: (`fig:TurbulenceHurricane`) A hurricane.

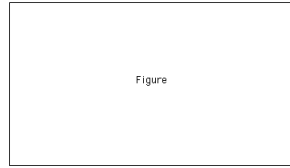


Figure 3.13: (**fig:LagrangianFramework**) Pictorial representation of a Lagrangian framework. Different independent initial conditions (x_o and y_o) yield different trajectories.

4. Climate change
5. Turbulence
6. Glassy materials
7. High-temperature superconductivity
8. Solar magnetism
9. Complexity
10. Consciousness

Viscosity, related to turbulence in a way, is also problematic. Viscosity, $\tau_{31} = \mu\dot{\gamma}$, is an empirical relationship that, in part, tries to account for the impact of turbulence at small scales. This is less of a problem in an Eulerian framework. (**Jim: Same comment as with Lagrangian comment earlier**).

GFD (geophysical fluid dynamics) is, basically, the special case of stratified fluids on a rotating sphere. Thermodynamics complicate things as well. Changes in internal energy from things like radiation and phase changes can have important dynamical implications.

3.6 Kinematics: Eulerian versus Lagrangian frameworks

Let's start with the word kinematics. Kinematics is the branch of mechanics dealing with the motion of bodies without worrying about the forces that cause the motion. Kinematics describes what the motion looks like. In terms of Newton's second law, $F = ma$, in kinematics you deduce F from a , where in dynamics you produce a from F .

Q1: The motion of solid bodies is traditionally described in a Eulerian framework. This is **false**. It is a chunk of something moving through space, that is, it is described in a Lagrangian framework.

Thinking about fluids in a Lagrangian sense is sometimes very useful, but in general we are not in a position to be able to describe the behavior of a fixed quantity of fluid as it evolves in time. However, we often are in the position of being able to describe what takes place as a fluid passes through a known volume of space.

3.6.1 Lagrangian

Let P denote the position of a system (see figure 3.13). At time t_o , the coordinates of the system are x_o and y_o . The position at $t > t_o$ is given by

$$\begin{aligned} x &= x(x_o, y_o, t) \\ y &= y(x_o, y_o, t) \end{aligned}$$

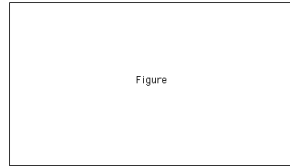


Figure 3.14: (`fig:SpacedVersusBunched`) Particles with an initially evenly space distribution can possibly yield areas of particle convergence.

The future position of P depends on the time and the initial position. Although x and y are dependent variables, x_o and y_o are independent variables. They define the future of the system. Different locations result in different trajectories.

The great thing about the Lagrangian approach is that velocity is just

$$\begin{aligned} u &= \frac{\partial x}{\partial t} \\ v &= \frac{\partial y}{\partial t} \\ w &= \frac{\partial z}{\partial t} \end{aligned}$$

and the acceleration is

$$\begin{aligned} a_x &= \frac{\partial u}{\partial t} = \frac{\partial^2 x}{\partial t^2} \\ a_y &= \frac{\partial v}{\partial t} = \frac{\partial^2 y}{\partial t^2} \\ a_z &= \frac{\partial w}{\partial t} = \frac{\partial^2 z}{\partial t^2} \end{aligned}$$

These expressions should look very familiar because the Lagrangian approach is used to describe dynamic behavior of solids.

Now consider a 2D fluid exhibiting interesting behavior. Seed the fluid with a bunch of particles (think of them as tiny blobs of fluid) and watch them evolve (see figure 3.14). Each particle has a velocity associated with it. As the particles evolve in time it is possible that there can be a net convergence. This type of situation leads to an illumination of the benefits and drawback of using a Lagrangian framework:

Benefit: In areas of particle convergence, one has lots of detailed information.

Drawback: In particle sparse areas, one does not know what is transpiring. It is important to remember that a lack of particles does not mean that nothing interesting is happening!

Benefit: Good estimates of $\Delta u/\Delta x$ (required in numerical models) in particle rich areas.

Drawback: Bad $\Delta u/\Delta x$ information in particle poor areas. Adding more particles is expensive.

Benefit: Easy to formulate linear momentum equation (and addressing some other concepts).

Drawback: Viscous stresses are difficult to model (hard to address some other concepts).

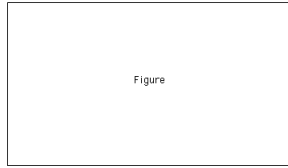


Figure 3.15: (`fig:EulerianFramework`) In an Eulerian framework, velocities are specified at points on a grid.

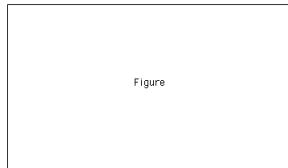


Figure 3.16: (`fig:EulerianTracerTimeEvolve1`) Eulerian model, $t = 4$ months.

3.6.2 Eulerian

In the Eulerian approach, we want to know what is happening at a given spatial location at a given time. The location, $P(x, y)$ is independently chosen. In the Eulerian view of the world, you specify velocity on a grid (at point P and elsewhere) as in figure 3.15. Instead of velocity of a point, here we are interested in the velocity at a point. Again, there are a number of benefits and drawbacks:

Benefit: Uniform coverage.

Drawback: “Low” resolution (a lot happens between points).

Benefit: Easy to get gradient information.

Drawback: More details require more points.

Benefit: “Easy” to handle viscosity.

Drawback: Nonlinear momentum equation.

For the most part, folks studying the atmosphere and ocean stick with Eulerian models. There is some Lagrangian stuff in stratospheric models where viscosity is low. That doesn’t mean the Lagrangian view isn’t interesting and/or useful in atmosphere and ocean models. It is good for post-processing for looking at tracers.

Show movies and pictures (figures 3.16 - 3.18) for examples.

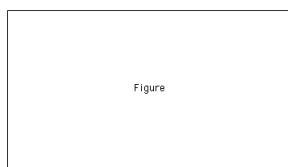


Figure 3.17: (`fig:EulerianTracerTimeEvolve2`) Eulerian model, $t = 8$ months.

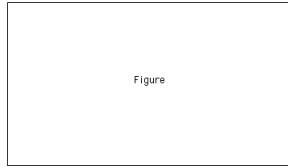


Figure 3.18: (fig:EulerianTracerTimeEvolve3) Eulerian model, $t = 12$ months.

Q2: Equations in a Lagrangian form have a fundamentally different meaning from those in Eulerian form. This is **false**. While the meaning is the same, we'll see in later lectures that Eulerian equations don't look much like their Lagrangian counterparts. But, when Lagrangian and Eulerian volumes overlap, the equations give identical answers.

3.6.3 Material Derivative

What this all means that when we are using Eulerian coordinates, we need to determine the same rate of change of “whatever” as we get from Lagrangian coordinates. Just like the Lagrangian case, we want to determine the rate of change following a particle. Consider some fluid quantity F where

$$F = f(x, y, z, t) \quad (3.17)$$

The fluid parcel moves over some Δt and changes

$$F + \Delta F = f(x + \Delta x, y + \Delta y, z + \Delta z, t + \Delta t) \quad (3.18)$$

We know that $\Delta x = u\Delta t$, $\Delta y = v\Delta t$, and $\Delta z = w\Delta t$, so

$$F + \Delta F = f(x + u\Delta t, y + v\Delta t, z + w\Delta t, t + \Delta t) \quad (3.19)$$

Using a Taylor expansion, and ignoring higher order terms we get

$$\begin{aligned} F + \Delta F &= f(x, y, z, t) + \left(u \frac{\partial f}{\partial x} + v \frac{\partial f}{\partial y} + w \frac{\partial f}{\partial z} + \frac{\partial F}{\partial t} \right) \Delta t \\ \frac{\Delta F}{\Delta t} &= u \frac{\partial f}{\partial x} + v \frac{\partial f}{\partial y} + w \frac{\partial f}{\partial z} + \frac{\partial F}{\partial t} \end{aligned} \quad (3.20)$$

Thus

$$\lim_{\Delta t \rightarrow 0} \frac{\Delta F}{\Delta t} = \frac{dF}{dt} \equiv \frac{DF}{Dt} = \left(\frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} \right) F \quad (3.21)$$

This is the material derivative

$$\begin{aligned} \frac{D}{Dt} &= \text{material derivative} \\ &= \left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \end{aligned} \quad (3.22)$$

Q3: $D/Dt = (\partial/\partial t + \mathbf{u} \cdot \nabla) =$ material derivative.

Q4: The temperature drop must be from advection.

3.7 Reading for class 4

KC01: 3.6 - 3.10

3.8 Notes for Jim

- It might be nice to have the quiz questions put into the notes to go along with the answers. If you pass on a copy of them, I will gladly do so.