Lecture 10

10.1 Administration

- None.

10.2 The (ideal, perfect, Euler) fluid

Consider the Navier-Stokes equation:

$$\rho \frac{D\mathbf{u}}{Dt} = \rho \mathbf{g} - \nabla p + \mu \nabla^2 \mathbf{u} + \frac{\mu}{3} \nabla(\nabla \cdot \mathbf{u}) \tag{10.1}$$

Assuming inviscid implies $\mu = 0$ and yields

$$\rho \frac{D\mathbf{u}}{Dt} = \rho \mathbf{g} - \nabla p \tag{10.2}$$

$$\frac{D\mathbf{u}}{Dt} = \mathbf{g} - \frac{1}{\rho} \nabla p \tag{10.3}$$

These are the "Euler's Equations" which describe an Euler, ideal, perfect fluid (all names for same thing). Not to be confused with a perfect gas. For the continuity equation, assuming incompressible ($\frac{D\rho}{Dt} = 0$) yields

$$\nabla \cdot \mathbf{u} = 0 \tag{10.4}$$

Next, lets expand the momentum equation:

$$\frac{D\mathbf{u}}{Dt} = \mathbf{g} - \frac{1}{\rho} \nabla p$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = \mathbf{g} - \frac{1}{\rho} \nabla p \tag{10.5}$$

Where

$$\mathbf{u} \cdot \nabla \mathbf{u} = \frac{1}{2} \nabla(\mathbf{u} \cdot \mathbf{u}) - \mathbf{u} \times (\nabla \times \mathbf{u}) \tag{10.6}$$

This is a vector invariant form in that it is invariant under coordinate transformations. Substituting this in yields

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{2} \nabla(\mathbf{u} \cdot \mathbf{u}) - \mathbf{g} + \frac{1}{\rho} \nabla p = \mathbf{u} \times (\nabla \times \mathbf{u}) \tag{10.7}$$

We can write $\mathbf{g}$ as $-\nabla gz$, and $\nabla \times \mathbf{u} = \omega$. Hence

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \left(\frac{1}{2} \mathbf{u} \cdot \mathbf{u} \right) + \nabla gz + \frac{1}{\rho} \nabla p = \mathbf{u} \times \omega \tag{10.8}$$

Note the gradients in the three terms on the LHS. Can almost pull a ∇ out, but $\frac{1}{\rho}$ gets in the way. We can deal with the $\frac{1}{\rho}$ by either (1) assuming ρ constant, or (2) $\rho = \rho(p)$ only **(Q1)**. Note that we've already assumed incompressible, that is ρ constant, so it needs to be said that the above work can be derived for an inviscid barotropic fluid. For (1)

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \left(\frac{1}{2} \mathbf{u} \cdot \mathbf{u} + gz + \frac{p}{\rho} \right) = \mathbf{u} \times \omega \tag{10.9}$$

For (2)

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \left(\frac{1}{2} \mathbf{u} \cdot \mathbf{u} + gz + \int \frac{dp}{\rho} \right) = \mathbf{u} \times \omega \tag{10.10}$$

Note that $\int \frac{dp}{\rho}$ is a perfect differential, that is, the integral is path independent – a conservative potential.

$$\Rightarrow \quad \frac{1}{\rho} \nabla p = \int \frac{dp}{\rho} \tag{10.11}$$

(Jim – This whole perfect differential stuff isn't immediately obvious to me, and the P's, p's, ρ's look quite similar in the photocopies; You may want to clean this up a bit). To see this, start with

$$P(p) = \int_{p_o}^{p(x,t)} \frac{dp}{\rho} \tag{10.12}$$

Then $\frac{\partial p}{\partial x} = \frac{1}{\rho(p)} \frac{dp}{dx} \Rightarrow \partial P = \frac{\partial p}{\rho(p)}$. Thus

$$\int \frac{dp}{\rho} = \int \frac{1}{\rho} \frac{dp}{d\rho} dp = \int \frac{dP}{dp} dp = \int dP = P \tag{10.13}$$

And we have that $\nabla P = \frac{1}{\rho} \nabla p$

10.3 Bernoulli function

Define

$$B = \frac{1}{2} \mathbf{u} \cdot \mathbf{u} + gz + \int \frac{dp}{\rho} \equiv \text{Bernoulli function} \tag{10.14}$$

Thus, the Bernoulli function form of the Euler equations become

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla B = \mathbf{u} \times \omega \tag{10.15}$$

There are three type of flow that that we can easily analyze with this form of the Euler equations

10.3.1 Steady flow

$$\frac{\partial \mathbf{u}}{\partial t} = 0 \quad \Rightarrow \quad \nabla B = \mathbf{u} \times \omega \tag{10.16}$$

This is a neat expression. ∇B is a vector normal to some surface where B is constant. The RHS is a vector perpendicular to both $\mathbf{u}$ and ω. This implies that B contains both streamlines (lines tangent to $\mathbf{u}$) and vortex lines (lines tangent to ω). Therefore B is constant along streamlines and vortex lines. This is seen in figure 10.1. This we have

$$\frac{1}{2} \mathbf{u} \cdot \mathbf{u} + gz + \int \frac{dp}{\rho} = \text{const. along streamlines and vortex lines} \tag{10.17}$$
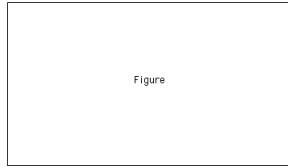
This is Bernoulli's equation **(Q2)**.

Figure 10.1: (`fig:Lec10SteadyFlowB`) Euler equation with Bernoulli function with steady flow. Here B is constant along streamlines and vortex lines. Streamlines and vortex lines can be at any arbitrary angle.

10.3.2   Steady irrotational flow

$$\frac{\partial \mathbf{u}}{\partial t} = 0, \ \mathbf{u} \times \omega = 0 \quad \Rightarrow \quad \nabla B = 0 \tag{10.18}$$

Thus

$$\frac{1}{2}\mathbf{u} \cdot \mathbf{u} + gz + \int \frac{dp}{\rho} = \text{constant everywhere} \tag{10.19}$$

Before going any further, let's talk about the implications of irrotational flow.

10.4   Irrotationalality and potential flow

First, why should we care about irrotational flow at all? As ocean and atmosphere folks, the answer is probably that we shouldn't; vorticity is hugely important in stratified fluids on a rotating sphere. However, it is used extensively in most other fields, typically for looking at flow around objects (wings, for example). As such, it is important that you know what it is and how to take advantage of it (and it is probably directly useful for you ocean engineer types).

Consider $\omega = \nabla \times \mathbf{u} = 0$ for irrotational flow. Recall from problem set 3 that

$$\nabla \cdot \omega = \nabla \cdot (\nabla \times \mathbf{u}) = 0 \tag{10.20}$$

More generally, the divergence of the curl of any vector function is zero where the mixed partials are equal. Well, it is also easy to show that the curl of the gradient of a scalar function is zero, that is

$$\nabla \times (\nabla \phi) = 0 \tag{10.21}$$

Again, this assumes the equality of the mixed partials. Try it! So, if we know $\nabla \times \mathbf{u} = 0$, and we know $\nabla \times (\nabla \phi) = 0$, then

$$\mathbf{u} = \nabla \phi \Rightarrow u = \frac{\partial \phi}{\partial x}, \ v = \frac{\partial \phi}{\partial y} \ w = \frac{\partial \phi}{\partial z} \tag{10.22}$$

Where ϕ is the velocity potential. This holds for any irrotational flow (in fact, the existence of ϕ is the sole criterion of irrotationality) and tells us that the three velocity components can be derived from a single variable, $\phi(x, y, z, t)$. Irrotational flow is called potential flow.

Thus, when we have an irrotational, incompressible fluid,

$$\nabla \cdot \mathbf{u} = 0 \Rightarrow \nabla \cdot \nabla \phi = 0 \Rightarrow \nabla^2 \phi = 0 \tag{10.23}$$

Therefore, the velocity potential obeys Laplace's equation. Laplace's equation is a classic problem in intro PDE classes. You might recall solving Laplace's equation on various geometries using separation of variables.

This $\mathbf{u} = \nabla\phi$ stuff looks a little like the relationship between velocity and streamfunction. remember the streamfunction? Remember the streamfunction from kinematics? If we have an incompressible flow, then

$$\nabla \cdot \mathbf{u} = 0 \tag{10.24}$$

We know we can satisfy this relationship be defining

$$\mathbf{u} = \nabla \times \mathbf{a} \tag{10.25}$$

since $\nabla \cdot (\nabla \times \mathbf{a}) = 0$. (Recall from before how the divergence of the curl of a vector function is zero). Because $\mathbf{a}$ is a vector, there is generally little to be gained taking this approach ($\mathbf{u}$ is a vector as well). But, if we have a 2D flow, then both the x-component and y-component of $\mathbf{a}$ are 0.

$$\mathbf{a} = 0\,\mathbf{i} + 0\,\mathbf{j} + \psi\,\mathbf{k} \tag{10.26}$$

$$\Rightarrow \quad \nabla \times \mathbf{a} = \frac{\partial\psi}{\partial y}\mathbf{i} - \frac{\partial\psi}{\partial x}\mathbf{j} \tag{10.27}$$

$$\Rightarrow \quad \nabla \cdot (\nabla \times \mathbf{a}) = \frac{\partial^2\psi}{\partial y \partial x} - \frac{\partial^2\psi}{\partial x \partial y} = 0 \tag{10.28}$$

So

$$u = \frac{\partial\psi}{\partial y}, \; v = -\frac{\partial\psi}{\partial x} \tag{10.29}$$

Where $\psi \equiv$ streamfunction. ψ is constant along a streamline. A check of this starts with

$$d\psi = \frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy \tag{10.30}$$

$$= -v\,dx + u\,dy \tag{10.31}$$

We know that on a streamline

$$\frac{dx}{u} = \frac{dy}{v} \quad \text{or} \quad \frac{dy}{dx} = \frac{v}{u} \tag{10.32}$$

which can only be satisfied if $d\psi = 0 \Rightarrow \psi = $ constant.

Thus in 2D a stream function can be defined when $\nabla \cdot \mathbf{u} = 0$. If the flow is irrotational, that is, $\nabla \times \mathbf{u} = 0$, and if $\nabla \cdot \mathbf{u} = 0$, this $\nabla^2\phi = 0$. **(Jim − This is what it looked like you were trying t say in your notes; is this how you want it?)**

10.5 Laplace's equations

If there is a flow that is 2D, incompressible, and irrotational, then

$$u = \frac{\partial\psi}{\partial y} = \frac{\partial\phi}{\partial x}, \; v = -\frac{\partial\psi}{\partial x} = \frac{\partial\phi}{\partial y} \tag{10.33}$$

(Q3). Note that ψ and ϕ are orthogonal. Thus,

$$\nabla \cdot \mathbf{u} = 0 \quad \Rightarrow \quad \nabla^2\phi = 0 \tag{10.34}$$

$$\nabla \times \mathbf{u} = 0 \quad \Rightarrow \quad \nabla^2\psi = 0 \tag{10.35}$$

$$\tag{10.36}$$

Again, these are the Laplace's equations. Laplace's equation represents, or should represent a familiar problem for you. The Dirichlet problem for the Laplace equation goes like

$$\nabla^2 \phi = a, \quad \phi = \phi_o \text{ on boundary} \tag{10.37}$$

The Neumann problem goes like

$$\nabla^2 \phi = a, \quad \phi' = \phi'_o \text{ normal to boundary on boundary} \tag{10.38}$$

They are well studied problems, with applications beyond fluids, and have well-behaved, stable, unique solutions. Again, in ocean and atmospheres we have little call for potential flow, but it is important you lot know it exists.

10.6 Aside on velocity potential

Why is it defined $\mathbf{u} = \nabla\phi$? This means that $\mathbf{u}$ moves up gradient! Why is it not defined $\mathbf{u} = -\nabla\phi$? Mathematically, this would be perfectly fine. To explain the sign convention, we need to bring in the stream function as well. We define it as

$$u = \frac{\partial\psi}{\partial y}, \ v = -\frac{\partial\psi}{\partial x} \tag{10.39}$$

This is another arbitrary sign choice. We could have said

$$u = -\frac{\partial\psi}{\partial y}, \ v = \frac{\partial\psi}{\partial x} \tag{10.40}$$

In fact, GFD folks tend to use this definition – more on this later. It turns out that these two sign choices are related. Recall Greg telling you about the Cauchy-Riemann conditions?

$$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}, \ \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x} \tag{10.41}$$

These equations allow us to write

$$w = \phi + i\,\psi \qquad \text{(the complex potential)} \tag{10.42}$$

which gives

$$\frac{dw}{dz} = u - i\,v \qquad \text{(the complex velocity)} \tag{10.43}$$

and permits one to use all the tools of complex functions to solve 2D fluids problems. If we are to define (in accordance to Kundu)

$$\mathbf{u} = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} = \nabla\phi \tag{10.44}$$

and

$$\mathbf{u} = \frac{\partial\psi}{\partial y}\mathbf{i} - \frac{\partial\psi}{\partial x}\mathbf{j} \tag{10.45}$$

then the Cauch-Riemann conditions are satisfied as we have

$$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}, \ \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x} \tag{10.46}$$
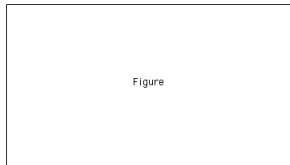
Figure 10.2: (`fig:Lec10GeostrophicPotentialFlow`) With the potential flow defined $\mathbf{u} = -\nabla\phi$ and $\mathbf{u} = -\frac{\partial\psi}{\partial y}\mathbf{i} + \frac{\partial\psi}{\partial x}\mathbf{j}$, the high and low ψ and p contours are in matching locations.

Rather, if we define (in accordance with Lamb)

$$\mathbf{u} = -\frac{\partial\phi}{\partial x}\mathbf{i} - \frac{\partial\phi}{\partial y}\mathbf{j} = -\nabla\phi \tag{10.47}$$

and

$$\mathbf{u} = -\frac{\partial\psi}{\partial y}\mathbf{i} + \frac{\partial\psi}{\partial x}\mathbf{j} \tag{10.48}$$

then the Cauch-Riemann conditions are still satisfied

$$-\frac{\partial\phi}{\partial x} = -\frac{\partial\psi}{\partial y}, \qquad -\frac{\partial\phi}{\partial y} = \frac{\partial\psi}{\partial x} \tag{10.49}$$

$$\Rightarrow \quad \frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}, \qquad \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x} \tag{10.50}$$

$$\tag{10.51}$$

Thus, in a sense, both definitions are "right" in that they both work. In fact, a mixed signs version could also be defined, but this is inconvenient from the Cauchy-Riemann perspective. So which is "better"? Probably $\mathbf{u} = -\nabla\phi$, $\mathbf{u} = -\frac{\partial\psi}{\partial y}\mathbf{i} + \frac{\partial\psi}{\partial x}\mathbf{j}$. Why?

a) Velocity flows down gradient.

b) Intuitive relatioship between ψ and p in geostrophic flow (see figure 10.2).

This explains the conventions in GFD books (although most never discuss $\mathbf{u} = -\nabla\phi$). (**Jim – This does not make sense to me. To you mean to say that most never discuss $\mathbf{u} = \nabla\phi$?**). Why does Kundu go the other way? Probably because Bachelor does it that way. Why does Bachelor do it that way? He derives $\mathbf{u} = \nabla\phi$ and states "there is no question here of an interpretation of ϕ as a potential energy function". Presumably this is because he invoked only maths to obtain $\mathbf{u} = \nabla\phi$, and there is no potential energy invoked. (**Jim – Hmmm... I do not think that "maths" is a word).**

10.7   Bernoulli form of the Euler equations

Back to the Bernoulli form of the Euler equations (**Q4**):

$$\frac{\partial\mathbf{u}}{\partial t} + \nabla B = \mathbf{u} \times \mathbf{omega} \tag{10.52}$$

If we assume irrotational flow, then $\mathbf{u} \times \omega = 0$, but we also have the opportunity to write $\frac{\partial\mathbf{u}}{\partial t}$ in the form of a gradient using the velocity potential. For irrotational flow, $\mathbf{u} = \nabla\phi$, and we have

$$\nabla\frac{\partial\phi}{\partial t} + \nabla B = 0 \tag{10.53}$$

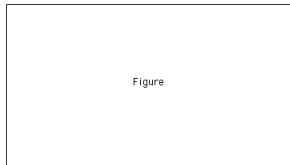$$\nabla\left(\frac{\partial\phi}{\partial t} + B\right) = 0 \tag{10.54}$$

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Figure 10.3: (`fig:Lec10LaminarBernoulli`) Classic example of irrotational, laminar flow analyzed using the Bernoulli function. As is seen, there is a pipe with a section with a small diameter (2) relative to either side of it (1) and (3). Following a streamline through the center of the pipe, $\rho\frac{1}{2}u_1{}^2 + p_1 = \rho\frac{1}{2}u_2{}^2 + p_2 = \rho\frac{1}{2}u_3{}^2 + p_3$. By inspection, $u_1 = u_3 < u_2$. For the Bernoulli function to be constant at each point, $p_1 = p_3 > p_2$.
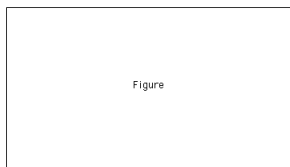

Figure 10.4: (`fig:Lec10WingBernoulli`) The flow around a wing can approximately be modeled as steady and irrotational. This implies that the Bernoulli function is constant everywhere. Thus, for the three points that are indicated on the figure, $\rho\frac{1}{2}u_1{}^2 + p_1 = \rho\frac{1}{2}u_2{}^2 + p_2 = \rho\frac{1}{2}u_3{}^2 + p_3$, assuming that the $\rho g\Delta z$ terms are small. Since the air over the top of the wing has a longer distance to travel, $u_1 = u_3 < u_2$ and hence $p_1 = p_3 > p_2$. Since $p_3 > p_2$, the lower pressure above the wing acts to "suck" of "hold" the wing up. (**Jim – Is this really how it should be modeled? The impression that I got from the spirited discussion that ensued from this example last year was that this was a really poor description of the system.**)

Integrate out the space derivatives, and you get

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\mathbf{u}\cdot\mathbf{u} + gz + \int \frac{dp}{\rho} = f(t) \tag{10.55}$$

This is Bernoulli's equation for irrotational flow, where f is spatially uniform at each t. This form of Bernoulli's equation is used to solve the gravity wave problem; pretty much the only problem in GFD that is irrotational.

Let's stay with irrotational Bernoulli, and look at an example in steady state (laminar, not turbulent). Further, let's assume $\rho = $ constant, so

$$\frac{1}{2}\mathbf{u}\cdot\mathbf{u} + gz + \frac{p}{\rho} = \text{constant} \tag{10.56}$$

Note: Bernoulli is really an expression of conservation of energy where the first term is kinetic, the second potential, and the third the energy change due to temperature or volume changes. A classic example is given in figure 10.3 and is corresponding caption. There is also an analogy for a wing as is given in figure 10.4.

It is best to think of Bernoulli as a kinematic expression here. (Kinematic, not dynamic; explains, but does not predict). If we know something about the velocity distribution, then we know something about the pressure distribution. Once we know the pressure distribution, we can say something about the forces on, for example, the wing. Similarly, if we know the pressure distribution, Bernoulli lets us say something about the velocity distribution.

Another example of the application of Bernoulli is the Pitot tube as seen in figure 10.5. Here again the flow is steady and irrotational, and ρ is constant. Hence, from comparing points 1 and
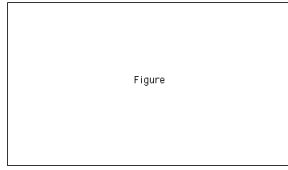
Figure 10.5: (`fig:Lec10PitotTubeBernoulli`) Diagram of Pitot tube configuration.

2 on the diagram, we have that $\rho\frac{1}{2}u_1{}^2 + p_1 = \rho\frac{1}{2}u_2{}^2 + p_2$. At point 2, $u_2 = 0$, so

$$\rho\frac{1}{2}u_1{}^2 + p_1 = p_2 \tag{10.57}$$

$$u_1 = \sqrt{\frac{2(p_2 - p_1)}{\rho}} \tag{10.58}$$

Note that this is exactly how airplanes determine their air speed. In this example, we can use the hydrostatic relation to get p_1 and p_2, in terms of Δz.

$$p_1 = \rho g z_1, \ p_2 = \rho g z_2$$
$$u_1 = \sqrt{2(gz_2 - gz_1)} = \sqrt{2g\Delta z} \tag{10.59}$$

Note that we can only do this when all streamlines at 1 are parallel between 1 and the upper wall.

There are a bunch of different pressures here:

$$\rho\frac{1}{2}u^2 + p \ \equiv \ \text{stagnation pressure} \tag{10.60}$$

$$\rho\frac{1}{2}u^2 \ \equiv \ \text{dynamic pressure} \tag{10.61}$$

$$p \ \equiv \ \text{static pressure} \tag{10.62}$$

$$\tag{10.63}$$

(Jim – You have some comment about the stagnation pressure on the RHS of page 11 of lecture 10 that I can not make out due to copying. Something like "pressure that would be reached if local flow is imagined to ???? to $u = 0$????").

10.8   Reading for class 11

KC01: 5.1-5.3, 5.7-5.8, 3.11 (review)